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Controlled Thermonuclear Processes

UNSTABLE PLASMA OSCILLATIONS IN A MAGNETIC FIELD

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OAK RIDGE NATIONAL LABORATORY
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ABSTRACT

Unstable oscillations of a uniform plasma in a constant uniform magnetic field are investigated using the Vlasov equations. It is found that if the velocity distributions of the electrons and ions are sufficiently anisotropic oscillations may occur whose amplitudes grow exponentially with time. Three different anisotropic distribution functions which lead to instabilities are studied. In one the electrons are all moving with the same velocity along the magnetic field and the ions are stationary. This case was previously considered by Buneman but without assuming the presence of a magnetic field. The second distribution function studied was one in which all electrons and ions move with the same speed perpendicular to the field. This case was previously considered by Malmfors, by Gross, and by Sen. This previous work is extended here. In the third distribution function considered the particles have a Maxwellian distribution of the velocity components perpendicular to the field. Provision is made in the function for varying the spread in the velocities in the direction of the field. If this spread in velocities along the field is sufficiently small unstable oscillations may occur.

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I. INTRODUCTION

Recent attempts to achieve controlled thermonuclear reactions have yielded experimental evidence which suggest the presence of a new type of instability. Thus, in a recent paper on the pinch effect Colgate and Furth[1] have reported evidence that small scale turbulence exist within the stabilized pinch configuration, and that this turbulence is responsible for decreasing the plasma conductivity, accelerating a small number of particles to high energies and greatly increasing the rate of heat transfer to the walls. The instability does not seem to be of the sort predicted by the hydromagnetic equations which involve a gross motion of the plasma to the walls of the container. These hydromagnetic instabilities are by now fairly well understood. Colgate and Furth suggest that the turbulence is due to plasma waves which are excited by runaway electrons.

W. Bernstein *et al.*[2] have reported a number of peculiar phenomena observed in Stellerator discharges. The phenomena include a decay of the discharge current in abrupt steps, the generation of intense non-thermal microwave noise and burst of x-rays due to loss of confinement of runaway electrons. These phenomena occured under conditions for which the plasma should not be subject to hydromagnetic instabilities.

The instabilities mentioned above bear some resemblance to an instability reported by Alfvén *et al.*[3] This instability was found in experiments on trochotrons. The main results may be summarized as follows. Electrons which are emitted in crossed electric and magnetic fields move in trochoidal paths and constitute a beam perpendicular to both fields. At low emission, i.e., at low electron density, the motion is in accordance with the motion calculated for single particles in external fields. At higher densities,

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1. S. A. Colgate and H. P. Furth, Science, 128, 337 (1958).
2. W. Bernstein, F. F. Chen, M. A. Heald, and A. Z. Kranz, Phys. Fluids 1, 430 (1958); Coor, Cunningham, Ellis, Heald and Kranz, Phys. Fluids 1, 411 (1958).
3. H. Alfvén, L. Lindberg, K. G. Malmfors, T. Wallmark, and E. Astrom, Kungl. Teckniska Hogskolans Handlingar NR 22 (1948).

however, the energy distribution of the electrons is rapidly changed in such a way that the electrons reach electrodes which are negative with respect to the cathode. At the same time the beam exhibits an abnormally strong noise. The transition between the normal and abnormal operating conditions occurs very sharply as the emission is increased. Malmfors[4] attempted to explain these results on the basis of unstable plasma oscillations. We shall review the theoretical work on this problem later in this paper.

Most investigations of a plasma in a magnetic field have relied on hydrodynamic equations, i.e., equations relating the densities, average velocities, pressures, and temperatures of the electrons and ions. In very high temperature and low density plasmas large departures from local thermodynamic equilibrium may be expected and the validity of hydrodynamic equations is questionable. Indeed, the calculations of Bernstein[5] indicate that a much wider variety of waves can exist in a plasma than the hydrodynamic equations predict. Bernstein's calculations are based on the Vlasov equations. (I.e., Boltzmann equations without collision terms for the electrons and ions plus Maxwell's equations.) It seems a reasonable guess that some of the experimentally observed instabilities which seem incomprehensible on the basis of the usual theories can be explained on the basis of the Vlasov equations. The present paper is devoted to exploring this possibility.

The development of this work is the following: In Section II we present the general formulation of the linearized problem. In doing this we follow fairly closely the work of Bernstein.[5] Unlike Bernstein we limit ourselves to the consideration of longitudinal oscillations. We do this in order to shorten the subsequent labor; we hope to consider transverse oscillations in a future publication. In Section III we consider a number of zeroth order distribution functions which lead to unstable plasma oscillations. In Section IV we discuss the applicability of our results and their limitations.

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4. Malmfors, Arkiv. Fys. 1, 569 (1950).
5. I. B. Bernstein, Phys. Rev. 109, 10 (1958).

II. THE LINEARIZED VLASOV EQUATIONS

The following set of equations is used

$$\frac{\partial f_i}{\partial t} + \vec{v} \cdot \frac{\partial f_i}{\partial \vec{r}} + \frac{e_i}{M_i} (\vec{E} + \frac{1}{c} \vec{v} \times \vec{B}) \cdot \frac{\partial f_i}{\partial \vec{v}} = 0 \tag{1}$$

$$\nabla \cdot \vec{E} = 4\pi \sum_i e_i \int f_i d^3v \tag{2}$$

$$\nabla \times \vec{E} = - \frac{1}{c} \frac{\partial \vec{B}}{\partial t} = 0 \tag{3}$$

In the above $f_i(\vec{r},\vec{v},t)$ is the distribution function for the i^{th} species of particle (electron or ion) and e_i and M_i are its charge and mass. $\vec{E}$ and $\vec{B}$ are the electric and magnetic field intensities. Equation 3 restricts our considerations to longitudinal waves. By Eq. 3 the magnetic field intensity is constant, and $\vec{E}$ can be written as

$$\vec{E} = - \nabla \Phi(\vec{r},t) \tag{4}$$

where Φ is a scalar potential. By proceeding in this way we have neglected the coupling between longitudinal and transverse waves. It can be shown that this is only valid if the phase velocity of the wave is much smaller than the velocity of light.

We now consider systems which depart only slightly from an equilibrium configuration in which $\vec{E} = 0$ and $f_i = f_i^0(\vec{v})$. We will take the uniform magnetic field to be in the z direction. It is easily seen that f_i^0 will satisfy Eq. 1 if

$$f_i^0 = f_i^0(v_\perp, v_z) \tag{5}$$

where

$$v_\perp^2 = v_x^2 + v_y^2 \tag{6}$$

Bernstein[5] assumed that f_i^o was the Maxwell distribution; we do not make that assumption. We write

$$f_i = f_i^o(v_\perp, v_z) + f_i^1(\vec{r}, \vec{v}, t) \tag{7}$$

and assume that f_i^1 and $\vec{E}$ are small quantities whose squares and products may be neglected. With the neglect of these small quantities Eq. 1 becomes

$$\frac{\partial f_i^1}{\partial t} + \vec{v} \cdot \frac{\partial f_i^1}{\partial \vec{r}} + \frac{e_i}{M_i c} (\vec{v} \times \vec{B}) \cdot \frac{\partial f_i^1}{\partial \vec{v}} = \frac{e_i}{M_i} \nabla \Phi \cdot \frac{\partial f_i^o}{\partial \vec{v}} \tag{8}$$

It is convenient to Fourier analyze in space and take Laplace transforms in time. We write

$$\bar{f}_i^1(\vec{k}, \vec{v}, t) = \int e^{-i\vec{k}\cdot\vec{r}} f_i^1(\vec{r}, \vec{v}, t) d^3r \tag{9}$$

and

$$\bar{\bar{f}}_i^1(\vec{k}, \vec{v}, P) = \int_0^\infty e^{-Pt} \bar{f}_i^1(\vec{k}, \vec{v}, t) dt \tag{10}$$

and similar expressions for the transforms of $\Phi(\vec{r}, t)$. When transformed, Eqs. 1 and 2 become

$$(P + i\vec{k}\cdot\vec{v})\bar{\bar{f}}_i^1 + \omega_{ci}\left(v_x \frac{\partial \bar{\bar{f}}_i^1}{\partial v_y} - v_y \frac{\partial \bar{\bar{f}}_i^1}{\partial v_x}\right) = \frac{ie_i}{M_i} \bar{\bar{\Phi}} k \cdot \frac{\partial f_i^o}{\partial \vec{v}} + \bar{g}_i(\vec{k}, \vec{v}) \tag{11}$$

and

$$k^2\bar{\bar{\Phi}} = 4\pi \sum_i e_i \int \bar{\bar{f}}_i d^3v \tag{12}$$

In the above $\omega_{ci} = \dfrac{e_i B}{M_i c}$ is the cyclotron frequency and

$$\bar{g}_i(\vec{k},\vec{v}) = \int e^{-i\vec{k}\cdot\vec{r}} f_i^1(\vec{r},\vec{v},0)d^3r \tag{13}$$

We now introduce the cylindrical coordinates $v_\perp$, ϕ, and v_z in velocity space and choose the vector $\vec{k}$ to lie in the xz plane. Then

$$\vec{v} = \vec{e}_1 v_\perp \cos\phi + \vec{e}_2 v_\perp \sin\phi + \vec{e}_3 v_z \tag{14}$$

$$\vec{k} = \vec{e}_1 k_\perp + \vec{e}_3 k_z \tag{15}$$

where $\vec{e}_1$, $\vec{e}_2$, and $\vec{e}_3$ are unit vectors along the x, y, and z axes. Equation 11 can now be written

$$\frac{\partial \bar{\bar{f}}_i^1}{\partial\phi} + \frac{1}{\omega_{ci}}(P + ik_\perp v_\perp \cos\phi + ik_z v_z)\bar{\bar{f}}_i^1$$
$$= \frac{ie_i}{M_i\omega_{ci}} \bar{\bar{\Phi}} \left\{ k_\perp \frac{\partial f_i^o}{\partial v} \cos\phi + k_z \frac{\partial f_i^o}{\partial v_z} \right\} + \frac{1}{\omega_{ci}} \bar{g}_i \tag{16}$$

Bernstein[5] has shown that the solution of Eq. 16 is given by

$$\bar{\bar{f}}_i^1 = \int_{\pm\infty}^{\phi} G(\phi,\phi') \left\{ \frac{ie_i}{M_i\omega_{ci}} \bar{\bar{\Phi}} \left[k_\perp \frac{\partial f_i^o}{\partial v_\perp} \cos\phi + k_z \frac{\partial f_i^o}{\partial v_z} \right] + \frac{1}{\omega_{ci}} \bar{g}_i \right\} d\phi' \tag{17}$$

where

$$G_i(\phi,\phi') = \exp\left[-\int_{\phi'}^{\phi}\left(\frac{P + ik_z v_z = ik_\perp v_\perp \cos\phi}{\omega_{ci}}\right) d\phi\right]$$

$$= \exp\left[-\left(\frac{P + ik_z v_z}{\omega_{ci}}\right)(\phi - \phi') - \frac{ik_\perp v_\perp}{\omega_{ci}}(\sin\phi - \sin\phi')\right] \tag{18}$$

Since $\bar{\bar{\Phi}}$ and f_i^o are independent of ϕ, Eq. 17 may be written

$$\bar{\bar{f}}_i^1 = \frac{ie_i}{M_i\omega_{ci}}\,\bar{\bar{\Phi}}\left\{k_\perp \frac{\partial f_i^o}{\partial v} I_{1i}(\phi) + k_z \frac{\partial f_i^{(0)}}{\partial v_z} I_{2i}(\phi)\right\} + \frac{1}{\omega_{ci}} S_i \tag{19}$$

where

$$I_{1i}(\phi) = \int_{\pm\infty}^{\phi} G_i(\phi,\phi') \cos\phi'\, d\phi' \tag{20}$$

$$I_{2i}(\phi) = \int_{\pm\infty}^{\phi} G_i(\phi,\phi')\, d\phi'$$

and

$$S_i = \int_{\pm\infty}^{\phi} G_i(\phi,\phi')\bar{g}_i\, d\phi' \tag{22}$$

Substituting Eq. 19 into Eq. 12 and solving for $\bar{\bar{\Phi}}$ gives

$$\bar{\bar{\Phi}} = \frac{-\sum_j \frac{4\pi e_j}{k^2 \omega_{cj}} \int S_j d^3v}{\left\{1 - \sum_j \frac{i4\pi e_j^2}{M_j \omega_{cj} k^2} \int d^3v \left\{k_\perp \frac{\partial f_j^o}{\partial v_\perp} I_{1j} + k_z \frac{\partial f_j^o}{\partial v_z} I_{2j}\right\}\right\}} \tag{23}$$

By use of the identity

$$e^{ix \sin\phi} = \sum_{n=-\infty}^{+\infty} J_n(x) e^{in\phi} \tag{24}$$

where J_n is Bessel function of order n, it may be shown that

$$\int_0^{2\pi} I_{1j}(\phi) d\phi = 2\pi \sum_{n=-\infty}^{+\infty} \frac{n J_n^2 \left(\frac{k_\perp v_\perp}{\omega_{cj}}\right)}{\left(\frac{k_\perp v_\perp}{\omega_{cj}}\right)\left(\frac{P + ik_z v_z}{\omega_{cj}} + in\right)} \tag{25}$$

and

$$\int_0^{2\pi} I_{2i}(\phi) d\phi = 2\pi \sum_{n=-\infty}^{+\infty} \frac{J_n^2 \left(\frac{k_\perp v_\perp}{\omega_{cj}}\right)}{\left(\frac{P + ik_z v_z}{\omega_{cj}} + in\right)} \tag{26}$$

Using Eqs. (25) and (26) we write Eq. (23) in the form

$$\bar{\bar{\Phi}} = \frac{- \sum_j \frac{4\pi e_j}{k^2 \omega_{cj}} \int S_i(\vec{v},k,P) d^3v}{1 - Y(P)} \tag{27}$$

where

$$Y(P) = 2\pi i \sum_j \frac{\omega_{Pj}^2}{k^2} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} dv_z \int_0^{\infty} v_\perp dv_\perp$$

$$\cdot \left\{ \frac{\omega_{cj}}{v_\perp} \frac{\partial f_i^o}{\partial v_\perp} \frac{n J_n^2\left(\frac{k_\perp v_\perp}{\omega_{cj}}\right)}{(P + ik_z v_z + in\omega_{cj})} + k_z \frac{\partial f_j^o}{\partial v_z} \frac{J_n^2\left(\frac{k_\perp v_\perp}{\omega_{cj}}\right)}{(P + ik_z v_z + in\omega_{cj})} \right\} \tag{28}$$

In Eq. 28 $\omega_{Pj} = \left(\frac{4\pi N_j e_j^2}{M_j}\right)$ is the plasma frequency. The normalization of f_j^o has been changed, so that its integral over all of velocity space is unity.

We are particularly interested in zeroth order distribution functions which cause the denominator of Eq. 27 to vanish for values of P which have a positive real part. That is

$$Y(P) = 1 \quad \text{for} \quad \mathrm{Re}(P) > 0 \tag{29}$$

If Eq. 29 is satisfied then there will exist plasma oscillations whose amplitudes increase exponentially with time. In the following section we will discuss several distribution functions which give rise to unstable plasma oscillations.

III. UNSTABLE DISTRIBUTION FUNCTIONS

We shall first consider the distribution functions

$$f_i^o = \delta(\vec{v}) = \frac{1}{2\pi} \frac{\delta(v_z)\delta(v_\perp)}{v_\perp} \tag{30}$$

for the ions, and

$$f_e^o = \frac{1}{2\pi} \frac{\delta(v_z - V)\delta(v_\perp)}{v_\perp} \tag{31}$$

for the electrons. In the above δ is the Dirac δ-function. These distribution functions correspond to zero temperature ions and electrons. The ions are at rest in the chosen coordinate system and the electrons have the velocity V along the magnetic field. Substitution of Eqs. 30 and 31 into Eq. 28 gives

$$Y(P) = -\left(\frac{\omega_{Pi}}{\omega_{ci}}\right)^2 \left[\left(\frac{k_\perp}{k}\right)^2 \frac{1}{\left(\frac{P}{\omega_{ci}}\right)^2 + 1} + \left(\frac{k_z}{k}\right)^2 \frac{1}{\left(\frac{P}{\omega_{ci}}\right)^2}\right]$$

$$-\left(\frac{\omega_{Pe}}{\omega_{ce}}\right)^2 \left[\left(\frac{k_\perp}{k}\right)^2 \frac{1}{\left(\frac{P + ik_zV}{\omega_{ce}}\right)^2 + 1} + \left(\frac{k_z}{k}\right)^2 \frac{1}{\left(\frac{P + ik_zV}{\omega_{ce}}\right)^2}\right] \tag{32}$$

We now consider some special cases of Eq. 32. If $k_\perp = 0$ it becomes

$$Y(P) = - \frac{\omega_{Pi}^2}{P^2} - \frac{\omega_{Pe}^2}{(P + ikV)^2} \tag{33}$$

This case has been considered by Buneman[6] who has shown that instabilities exist for a proper choice of ω_{Pi}, ω_{Pe}, and kV. We shall use this example to introduce the use of the Nyquist[7] diagram in determining whether the system is stable or unstable. Let $P = \gamma + i\omega$ where γ is a constant and ω varies from $-\infty$ to $+\infty$. Then

$$Y(\gamma + i\omega) = \frac{\omega_{Pi}^2}{(\omega - i\gamma)^2} + \frac{\omega_{Pe}^2}{(\omega + kV - i\gamma)^2} \tag{34}$$

defines a mapping of the curve

$$Re(P) = \gamma = \text{constant} \tag{35}$$

in the P-plane onto a curve in the Y(P)-plane which has the general shape shown in Fig. 1. It is seen from Eq. 34 that

$$Y(\gamma + i\omega) = Y^*(- \gamma + i\omega) \tag{36}$$

so that the map of

$$Re(P) = - \gamma = \text{constant} \tag{37}$$

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6. O. Buneman, Phys. Rev. Letters, 1, 8 (1958).
7. H. M. James, N. B. Nichols, R. S. Philips. Theory of Servomechanism (MIT Radiation Laboratory Series, McGraw-Hill, New York, 1947) p. 70.

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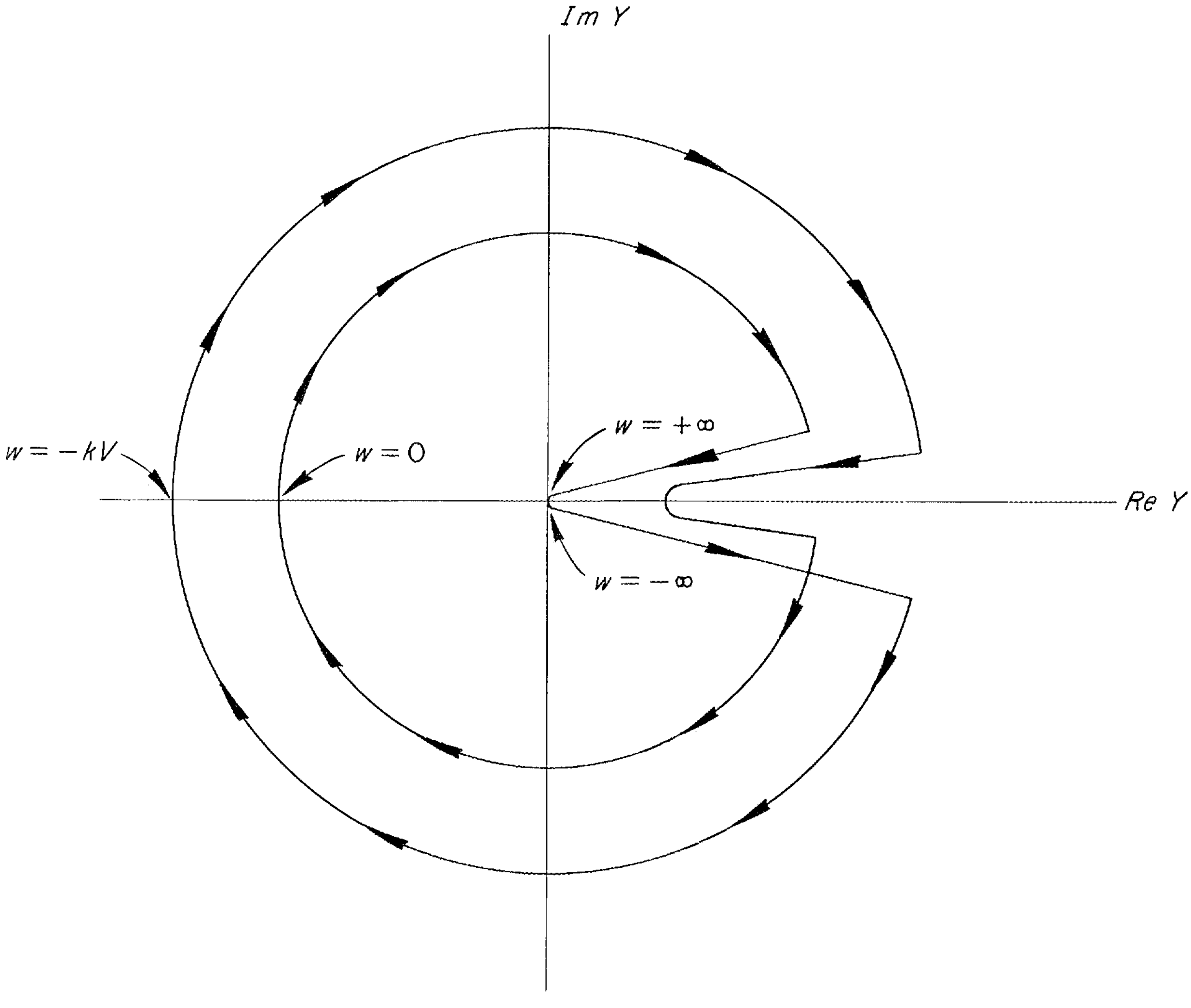

Fig. 1. Nyquist Diagram Corresponding to Eq. (34).

may be obtained from Fig. 1 by reflection in the real axis. Now it may easily be seen that points that lie within the strip $-\gamma < \mathrm{Re}(P) < +\gamma$ map onto the region exterior to the curve and points outside of this strip map onto the interior of the curve. Therefore, if the point +1 is enclosed by the curve, there will exist a value of P with $\mathrm{Re}(P) > \gamma$ which satisfies

$$Y(P) = 1 \tag{38}$$

and consequently there are unstable oscillations. There will also be another value of P with $\mathrm{Re}(P) < -\gamma$ which satisfies Eq. 38 because of Eq. 36. The same result may be obtained if the curve $Y(i\omega)$ is sketched as in Fig. 2. Now

$$Y(i\omega) = +1 = \frac{\omega_{Pi}^2}{\omega^2} + \frac{\omega_{Pe}^2}{(\omega + kV)^2} \tag{39}$$

is an algebraic equation of the fourth degree for ω. Since the coefficients are real complex roots must appear as conjugate pairs. From Fig. 2 it may be seen that for some values of ω_{Pi}, ω_{Pe}, and kV the horizontal line labeled + 1 will intersect the curve $Y(i\omega)$ four times corresponding to four real roots of Eq. 39. For other values of these parameters the intersections between $\omega = 0$ and $\omega = kV$ will disappear and two of the roots must be complex.

We next consider the special case $k_z = 0$. Equation 32 becomes

$$Y(P) = -\frac{\omega_{Pi}^2}{P^2 + \omega_{ci}^2} - \frac{\omega_{Pe}^2}{P^2 + \omega_{ce}^2} = 1 \tag{40}$$

A plot of the Nyquist diagram or of $Y(i\omega)$ immediately shows that all four roots of Eq. 40 are imaginary and consequently there are no unstable modes for $k_z = 0$. If neither $k_\perp$ nor k_z is zero there will eight roots of $Y(P) = 1$ and two of them may be complex depending of the values of ω_{Pi}, ω_{Pe}, ω_{ci}, ω_{ce}, and kV.

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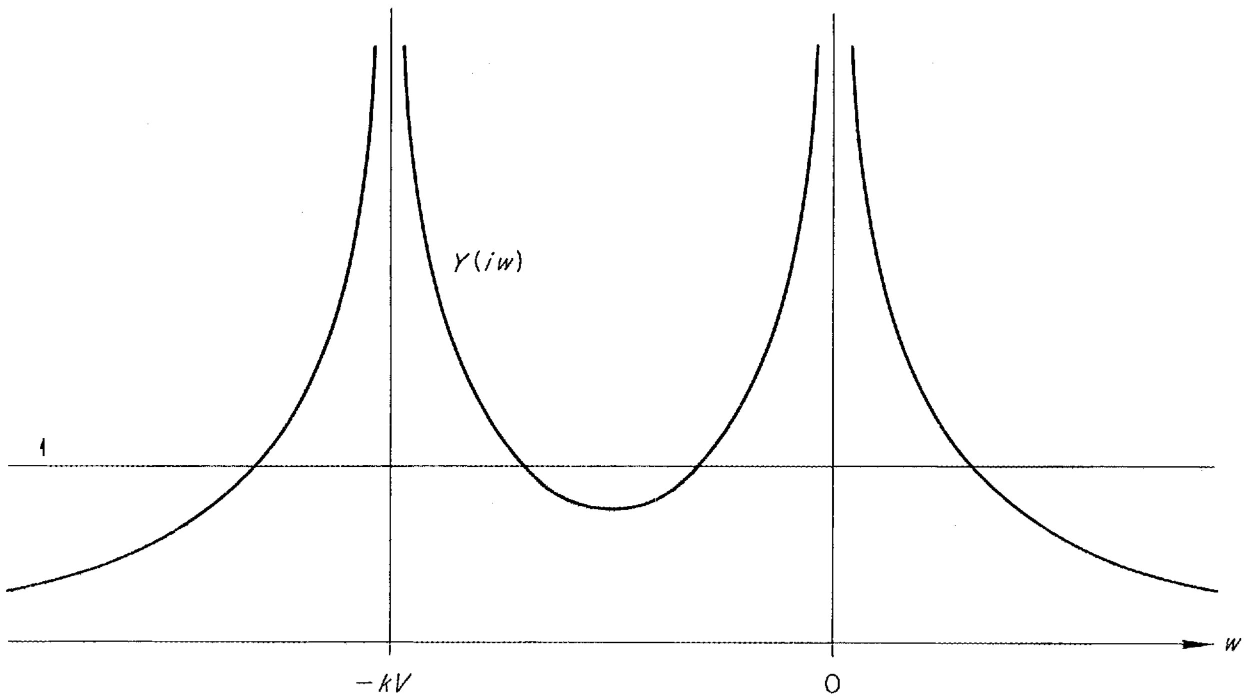

Fig. 2. $Y(iw)$ as Given by Eq. (39).

We next consider the distribution functions

$$f_i^o = \frac{1}{2\pi}\,\delta(v_z)\,\frac{\delta(v_\perp - V_i)}{v_\perp} \tag{41}$$

$$f_e^o = \frac{1}{2\pi}\,\delta(v_z)\,\frac{\delta(v_\perp - V_e)}{v_\perp} \tag{42}$$

These distributions correspond to all of the particles having their velocity vectors in the plane perpendicular to the magnetic field and all of the ions having the speed V_i and all of the electrons having the speed V_e. Substituting Eqs. 41 and 42 into Eq. 28 and integrating by parts gives the dispersion relation

$$1 = Y(P) = \sum_j \frac{\omega_{Pj}^2}{\omega_{cj}^2} \sum_{n=-\infty}^{+\infty} \left\{ - i\left(\frac{k_\perp}{k}\right)^2 \frac{n}{\left(\frac{P}{\omega_{cj}} + in\right)} \left[\frac{1}{b_j}\,\frac{d}{db_j}\,J_n^2(b_j)\right] - \left(\frac{k_z}{k}\right)^2 \frac{1}{\left(\frac{P}{\omega_{cj}} + in\right)^2} \left[J_n^2(b_j)\right] \right\} \tag{43}$$

where $b_j = \frac{k_\perp V_j}{\omega_{cj}}$.

We shall first of all neglect the ion motion and set $k_z = 0$. Equation 43 then becomes

$$1 = Y(P) = - i\,\frac{\omega_{Pe}^2}{\omega_{ce}^2} \sum_{n=-\infty}^{+\infty} \frac{n}{\left(\frac{P}{\omega_{ce}} + in\right)}\,\frac{1}{b_e}\,\frac{d}{db_e}\left(J_n^2(b_e)\right) \tag{44}$$

Equation 44 is the dispersion relation found by Gross.[8] A similar relation had previously been found by Malmfors[4] using the distribution function of Eq. 42 and on the basis of this Malmfors predicted instabilities. However, Gross found an error which invalidated Malmfors conclusions. Gross examined Eq. 44 for small values of b_e and found that there were no unstable roots. He conjectured that there were no unstable roots for any value of b_e. Later Sen[9] rederived Eq. 44 and numerically found unstable roots for large b_e. We shall show that there are unstable roots for values of b_e greater than 1.84 which is the value at which $J_1(b_e)$ has its first maximum. There no unstable roots for smaller values of b_e.

In Fig. 3 we have sketched a part of the Nyquist diagram for $b_e < 1.84$ and in Fig. 4 we have done the same for $b_e > 1.84$. The difference in the diagrams is due to the change in sign of the derivative of $J_1^2(b_e)$ when b_e becomes greater than 1.84. It is seen that the curve in Fig. 4 can enclose the point + 1 when the parameters are suitably chosen

If we set $k_\perp = 0$ in Eq. 43 we find

$$1 = Y(P) = -\frac{1}{P^2}(\omega_{Pi}^2 + \omega_{Pe}^2) \tag{45}$$

from which

$$P = \pm i(\omega_{Pi}^2 + \omega_{Pe}^2)^{\frac{1}{2}} = \pm i\omega_{Pe}\left(1 + \frac{m}{M}\right)^{\frac{1}{2}} \tag{46}$$

The roots given by Eq. 46 correspond to stable plasma oscillations along the lines of the magnetic field.

When neither k_z nor $k_\perp$ vanish, it may be seen from Eq. 43 that there can be instabilities even for $b < 1.84$. This possibility was not considered

8. E. P. Gross, Phys. Rev. 82, 232 (1951).
9. H. K. Sen, Phys. Rev. 88, 816 (1952).

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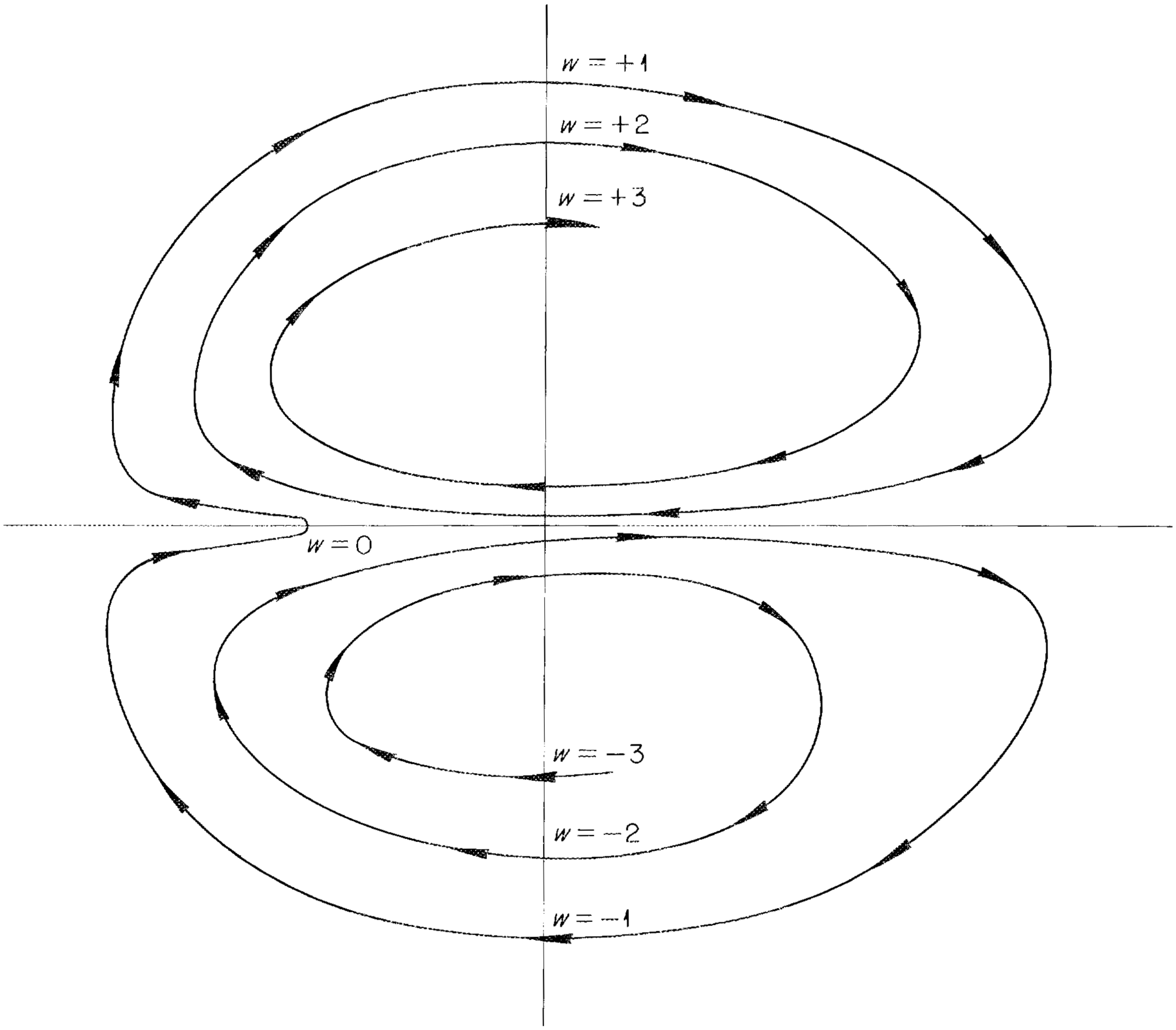

Fig. 3. Nyquist Diagram Corresponding to Eq. (44) with $b_e < 1.84$.

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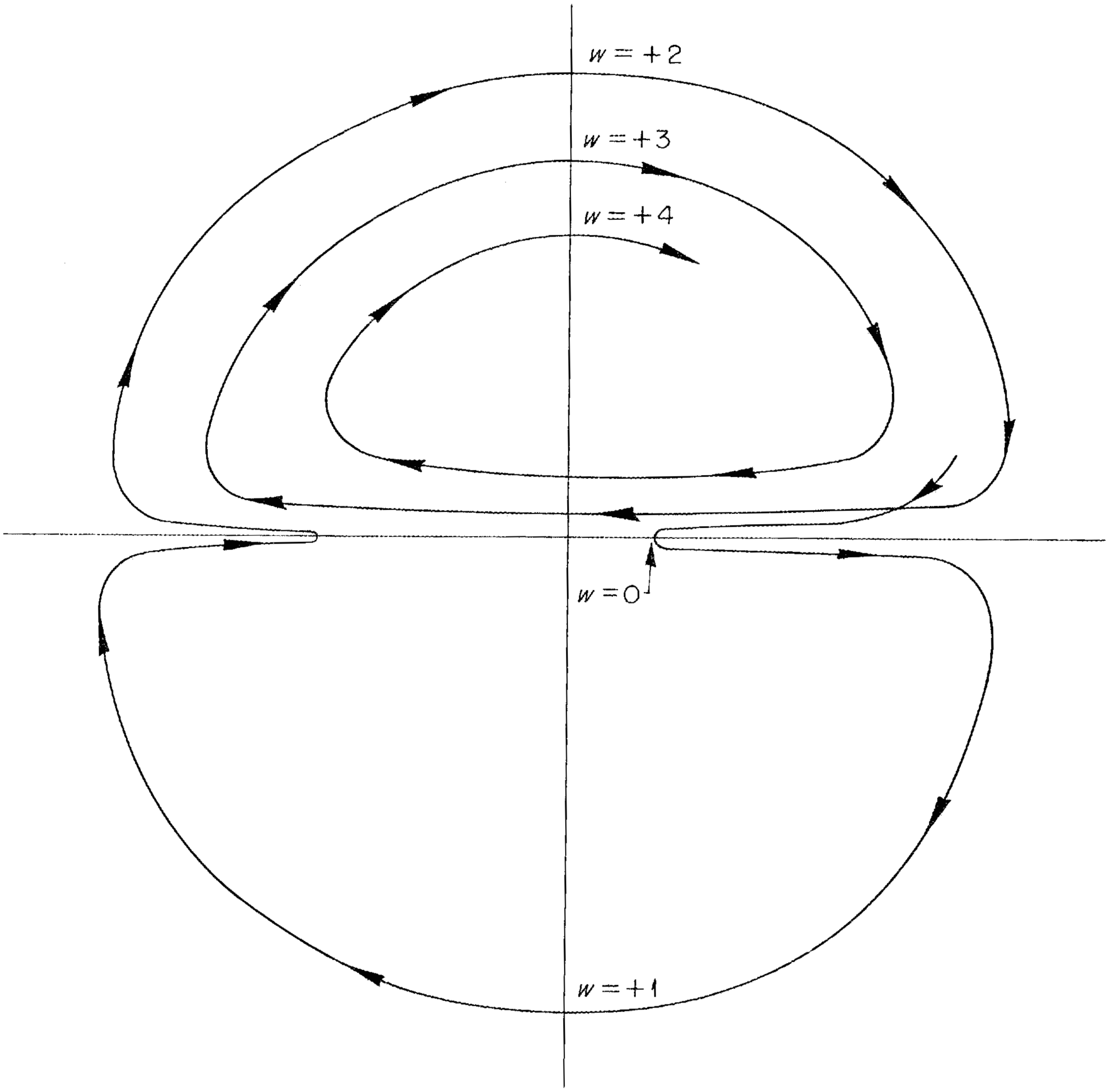

Fig. 4. Nyquist Diagram Corresponding to Eq. (44) with $b_e > 1.84$.

by Gross or Sen who restricted their discussions to waves propagating perpendicular to the magnetic field. The analysis of this case is very similar to the one which we shall now discuss.

We now wish to consider the effect of a spread of velocities on the stability. For that purpose we consider the distribution function

$$f_i^o = \frac{1}{2\pi}\left(\frac{2}{\alpha_{\perp i}^2}\right)\left(\frac{\alpha_{zi}}{\pi}\right)\frac{e^{-v_\perp^2/\alpha_{\perp i}^2}}{v_z^2+\alpha_{zi}^2} \tag{47}$$

and a similar distribution function for the electrons. The parameters $\alpha_\perp$ and α_z measure the spread in velocity in the directions perpendicular and parallel to the field. The function given by Eq. 47 was chosen because it resembles the Maxwell-Boltzmann function and allows the integrals in Eq. 28 to be evaluated in terms of known functions.

Substituting Eq. 47 into Eq. 28 and using the formula[10]

$$\int_0^\infty v_\perp e^{-v_\perp^2/\alpha_\perp^2} J_n^2\left(\frac{k_\perp v_\perp}{\omega_c}\right) dv_\perp = \frac{\alpha_\perp^2}{2} e^{-\alpha_\perp^2 k_\perp^2/2\omega_c^2} I_{|n|}\left(\frac{\alpha_\perp^2 k_\perp^2}{2\omega_c^2}\right), \tag{48}$$

one finds

$$Y(P) = \sum_j \omega_{Pj}^2 \sum_{n=-\infty}^{+\infty} e^{-\lambda_j} I_{|n|}(\lambda_j)\left\{- i\,\frac{1}{\lambda_j}\left(\frac{k_\perp}{k}\right)^2 \frac{\dfrac{n}{\omega_{cj}}}{(P + |k_z|\alpha_{zj} + in\omega_{cj})} - \left(\frac{k_z}{k}\right)^2 \frac{1}{\left[P + |k_z|\alpha_{zj} + in\omega_{cj}\right]^2}\right\} \tag{49}$$

In Eq. 49 $\lambda_j = \left(\dfrac{\alpha_{\perp j}^2 k_\perp^2}{2\,\omega_{cj}^2}\right)$.

10. G. N. Watson, Theory of Bessel Functions (Cambridge University Press, Cambridge, Massachusetts, 1945), p. 395.

We shall first consider the special case in which $k_\perp = 0$ and the motion of the ions can be neglected. We find for the dispersion relation

$$1 = Y(P) = - \frac{\omega_{Pe}^2}{(P + |k_z| \alpha_{ze})^2} \tag{50}$$

from which

$$P = - |k_z| \alpha_{ze} \pm i\omega_{Pe} \tag{51}$$

which corresponds to damped oscillations at the plasma frequency. The physical origin of the damping may be seen as follows: The oscillations decay in a time of the order of

$$\tau = \frac{1}{k_z \alpha_z} = \frac{\lambda}{2\pi\alpha_z} \tag{52}$$

Since α_z is a typical thermal velocity in the z direction, τ is the time required for a typical particle to move a distance $\lambda/2\pi$ by reason of its thermal motion. Due to thermal motion the particles get out of phase with the oscillation. Damping of this sort was first discovered by Landau[11] who assumed a Maxwellian distribution and consequently found a decay time different from that given by Eq. 52.

Still neglecting the motion of the ions we consider the case $k_z = 0$. The dispersion relation becomes

$$1 = Y(P) = - \frac{i\,\omega_{Pe}^2}{\omega_{ce}\lambda_e} e^{-\lambda_e} \sum_{n=-\infty}^{+\infty} I_{|n|}(\lambda_e) \frac{n}{(P + in\omega_{ce})} \tag{53}$$

11. L. Landau, J. Exptl.-Theoret. Phys. (USSR) 1, 574 (1946).

Equation 53 may be shown to reduce to the expression found by Bernstein.[2] There are an infinite number of roots of Eq. 53 all of which are imaginary; hence there is neither instability nor Landau damping for waves propagating perpendicular to the field.

We now return to Eq. 49 and take into account the motion of both electrons and ions. For the time being we will let $\alpha_z = 0$. We find it convenient to let $P = i\omega$ and write Eq. 49 in the form

$$
\begin{aligned}
1 = Y(i\omega) = \omega_{Pi}^2 \Bigg\{ & e^{-\beta_i n_\perp^2} I_o(\beta_i n_\perp^2) \frac{n_z^2}{\omega^2} + 2\, e^{-\beta_i n_\perp^2} \sum_{n=1}^{\infty} I_n(\beta_i n_\perp^2) \\
& \left[\frac{1}{\beta_i} \frac{n^2}{(\omega^2 - n^2\omega_{ci}^2)} + n_z^2 \frac{\omega^2 + n^2\omega_{ci}^2}{(\omega^2 - n^2\omega_{ci}^2)^2} \right] \Bigg\} + \omega_{Pe}^2 \Bigg\{ e^{-\beta_e n_\perp^2} I_o(\beta_e n_\perp^2) \frac{n_z^2}{\omega^2} \\
& + 2e^{-\beta_e n_\perp^2} \sum_{n=1}^{\infty} I_n(\beta^e n_\perp^2) \left[\frac{1}{\beta_e} \frac{n^2}{(\omega^2 - n^2\omega_{ce}^2)} + n_z^2 \frac{\omega^2 + n^2\omega_{ce}^2}{(\omega^2 - n^2\omega_{ce}^2)^2} \right] \Bigg\}
\end{aligned}
\tag{54}
$$

where

$$
\beta_i = \frac{\alpha_{\perp i}^2 k^2}{2\omega_{ci}^2} \tag{55}
$$

$$
\beta_e = \frac{\alpha_{\perp e}^2 k^2}{2\omega_{ce}^2} \tag{56}
$$

$$
n_\perp = \frac{k_\perp}{k} \tag{57}
$$

$$n_z = \frac{k_z}{k} \tag{58}$$

It may be seen from Eq. 54 that $Y(i\omega)$ has singularities whenever ω is an integral multiple of the ion or electron cyclotron frequency. When $\omega \ll \omega_{ce}$ we may neglect the terms containing $I_n(\beta_e n_\perp^2)$ for $n > 1$ and write

$$1 = \frac{\omega_{Pe}^2}{\omega_{ci}^2} \left\{ \left[e^{-\beta_e n_\perp^2} I_o(\beta_e n_\perp^2) + \frac{M_e}{M_i} e^{-\beta_i n_\perp^2} I_o(\beta_i n_\perp^2) \right] \frac{n_z^2}{\omega'^2} + 2 \frac{M_e}{M_i} e^{-\beta_i n_\perp^2} \sum_{n=1}^{\infty} I_n(\beta_i n_\perp^2) \left[\frac{1}{\beta_i} \frac{n^2}{(\omega'^2 - n^2)} + n_z^2 \frac{\omega'^2 + n^2}{(\omega'^2 - n^2 \omega_{ci}^2)^2} \right] \right\} \tag{59}$$

where $\omega' = \omega/\omega_{ci}$ and we have used

$$\frac{\omega_{Pi}^2}{\omega_{Pe}^2} = \frac{M_e}{M_i} \tag{60}$$

We will now use the smallness of the mass ratio to find approximate roots of Eq. 59. We multiply both sides of the equation by $\omega'^2(\omega'^2 - \ell^2)^2$ where ℓ is an integer and obtain

$$\left[\omega'^2 - n_z^2 \frac{\omega_{Pe}^2}{\omega_{ci}^2} A \right] (\omega'^2 - \ell^2)^2 = 2 \frac{M_e}{M_i} \frac{\omega_{Pe}^2}{\omega_{ci}^2} e^{-\beta_i n_\perp^2} \sum_{n=1}^{\infty} I_n(\beta_i n_\perp^2) \left[\frac{n^2}{\beta_i} \frac{(\omega'^2 - \ell^2)^2}{(\omega'^2 - n^2)} + n_z^2 \frac{(\omega'^2 - \ell^2)^2 (\omega'^2 + n^2)}{(\omega'^2 - n^2)^2} \right] \tag{61}$$

where

$$A = e^{-\beta_e n_\perp^2} I_o(\beta_e n_\perp^2) + \frac{M_e}{M_i} e^{-\beta_i n_\perp^2} I_o(\beta_i n_\perp^2) \tag{62}$$

Now if M_e/M_i were zero Eq. 61 would have the roots

$$\omega' = \frac{\omega}{\omega_{ci}} = \pm \ell \tag{63}$$

and

$$\omega' = \frac{\omega}{\omega_{ci}} = \pm n_z \frac{\omega_{Pe}}{\omega_{Pi}} A^{1/2} \tag{64}$$

This suggests that in order to find an approximation to the roots near $\omega' = \ell$ we substitute ℓ for ω' everywhere in Eq. 61 except in the factor $(\omega'^2 - \ell^2)^2$ on the left hand side. This gives

$$\omega'^2 = \ell^2 + \frac{\frac{\omega_{Pe}}{\omega_{ci}} \left[4 \frac{M_e}{M_i} e^{-\beta_i n_\perp^2} I_\ell(\beta_i n_\perp^2) n_z^2 \ell^2\right]^{1/2}}{\left[\ell^2 - n_z^2 \frac{\omega_{Pe}^2}{\omega_{ci}^2} A\right]^{1/2}} \tag{65}$$

$$\omega' = \ell \left[1 + \frac{2n_z \left(\frac{\omega_{Pe}}{\omega_{ci}}\right) \left[\frac{M_e}{M_i} e^{-\beta_i n_\perp^2} I_\ell(\beta_i n_\perp^2)\right]^{1/2}}{\ell \left[\ell^2 - n_z^2 \frac{\omega_{Pe}^2}{\omega_{ci}^2} A\right]^{1/2}}\right]^{1/2} \tag{66}$$

or approximately

$$\omega' = \pm \ell \pm \frac{n_z \left(\frac{\omega_{Pe}}{\omega_{ci}}\right) \left[\frac{M_e}{M_i} e^{-\beta_i n_\perp^2} I_\ell(\beta_i n_\perp^2)\right]^{1/2}}{\left[\ell^2 - n_z^2 \frac{\omega_{Pe}^2}{\omega_{ci}^2} A\right]^{1/2}} \tag{67}$$

or finally

$$\omega = \pm \ell\omega_{ci} \pm \frac{n_z \omega_{Pe} \left[\frac{M_e}{M_i} e^{-\beta_i n_\perp^2} I_\ell(\beta_i n_\perp^2)\right]^{1/2}}{\left[\ell^2 - n_z^2 \frac{\omega_{Pe}^2}{\omega_{ci}^2} A\right]^{1/2}} \tag{68}$$

Equation 68 is an approximate expression for the roots of Eq. 59 in the neighborhood of $\omega = \pm \ell\omega_{ci}$. The approximations that have been made are valid only when the second term in Eq. 68 is much smaller than the first. It is seen that ω becomes complex indicating an instability whenever

$$n_z^2 \frac{\omega_{Pe}^2}{\omega_{ci}^2} A > \ell^2 \tag{69}$$

In the limit

$$n_z^2 \frac{\omega_{Pe}^2}{\omega_{ci}^2} A >> \ell^2 \tag{70}$$

Equation 68 becomes

$$\omega = \pm \ell\omega_{ci} \pm i\omega_{ci} \left[\frac{M_e}{M_i} e^{-\beta_i n_\perp^2} I_\ell(\beta_i n_\perp^2)\right]^{1/2} \tag{71}$$

The most unstable mode will be the one corresponding to $\ell = 1$; for this mode

$$\omega = \pm \omega_{ci} \pm i \omega_{ci} \left[\frac{M_e}{M_i} e^{-\beta_i n_\perp^2} I_1(\beta_i n_\perp^2) \right]^{1/2} \tag{72}$$

We can estimate the rate of growth of the most unstable mode for two cases. In the first we give

$$e^{-\beta_i n_\perp^2} I_1(\beta_i n_\perp^2)$$

Its maximum value of 0.219 which makes

$$\omega = \pm \omega_{ci} \pm i \omega_{ci} \left[0.219 \frac{M_e}{M_i} \right]^{1/2} \tag{73}$$

or

$$\omega \approx \pm \omega_{ci} \pm i \frac{\omega_{ci}}{120} \tag{74}$$

for deuterium. In the second case we take $\beta_i n_\perp^2$ very small, then

$$e^{-\beta_i n_\perp^2} I_1(\beta_i n_\perp^2) \approx \frac{1}{2} \beta_i n_\perp^2 \frac{\alpha_{\perp i}^2 k_\perp^2}{4 \omega_{ci}^2} \tag{75}$$

and

$$\omega \approx \pm \omega_{ci} \pm i \frac{1}{2} \alpha_\perp k_\perp \sqrt{\frac{M_e}{M_i}} \tag{76}$$

The effect of a finite α_z can be taken into account now just by replacing ω by $\omega - i\alpha_z k_z$ in the preceding equations. This indicates that a spread of velocities in the direction of the field causes a damping of the oscillations. Probably no quantitative significance should be attached to $\alpha_z k_z$ since the distribution in v_z given by Eq. 47 is a rather unrealistic one. We have already remarked that the decay time found by Landau using a Maxwellian distribution was not the same as our Eq. 52.

A criterion for instability can be found from Eq. 69. In general n_z^2 and A are close to unity and the most unstable mode is $\ell = 1$. Replacing n_z^2, A, and ℓ by unity gives the instability criterion

$$\omega_{Pe}^2 > \omega_{ci}^2 \tag{77}$$

$$\frac{4\pi Ne^2}{M_e} > \frac{e^2B^2}{M_i^2c^2} \tag{78}$$

or

$$N > \left(\frac{M_e}{M_i}\right) \frac{B^2}{4\pi M_i c^2} \tag{79}$$

IV. DISCUSSION

In the preceding section we discussed several distribution functions which give rise to unstable plasma oscillations. The instabilities which arise when there is relative motion between electrons and ions have been discussed by Buneman for the case of no magnetic field and will not be discussed further here.

The principle result of this work is that instabilities may arise when the velocity distributions are flattened in the direction of the magnetic field. If we take $B = 10^4$ gauss and M_i to be the deuteron mass, Eq. 79 predicts that instabilities will develop when the particle densities exceed about 10^7. This is a particle density much below any proposed for thermonuclear machines known to the author. The e-folding times calculated from Eq. 73 or Eq. 74 are, of course, much shorter than containment times necessary for a practical device.

These instabilities are reduced by Landau damping if there is a spread of velocities along the direction of the field. However, this damping is small for waves with small k_z. We have shown for the distribution function given by Eq. 47 that there is no instability if $k_z = 0$.

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