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Controlled Thermonuclear Processes

COHERENT RADIATION FROM A PLASMA

E. G. Harris
A. Simon

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Abstract

The so-called "Collisionless Boltzmann Equation" or "Vlasov Equation" has been derived previously by Harris by use of a complete statistical treatment of both the plasma particles and the electromagnetic fields. It is shown that a consequence of this approximation is that the entropy of the electromagnetic field as well as that of the plasma particles is a constant. This result is used to demonstrate that only completely coherent radiation may be emitted by a plasma obeying the Vlasov equation.

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I. Introduction

The Vlasov equation or "Collisionless Boltzmann equation" have been derived previously by Harris[1] by starting from the Liouville equation and using a complete statistical treatment of both the plasma particles and the electromagnetic field. One then integrates over the coordinates of all but one particle or all but one field oscillator and assumes that the pair distribution function is factorable into a product of one particle or one oscillator distribution functions. The resulting equation for the one particle distribution function is easily seen to be exactly the usual Vlasov equation. The corresponding equation for the one oscillator distribution function has some interesting implications which have not been considered previously.

In particular, one can define an entropy for the electromagnetic field and show that it is a constant in time. It is quite easy to show that the entropy of a radiation field will be the same as it is in the absence of radiation only if the field is completely coherent. Hence a plasma obeying the Vlasov equation can only radiate coherently if there is initially no radiation present.

The one particle and one oscillator distribution equations are derived in the next section. We then define the particle and field entropies and show that these are conserved in time. Finally, it is demonstrated that any incoherence raises the entropy of the field.

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1. E. G. Harris, NRL Report 4944, May 17, 1957.

II. The Equations for the One-Particle and One-Oscillator Distribution Functions

Using the notation of Heitler,[2] we expand the vector potential in a series of orthogonal vector functions which are periodic on some surface bounding our systems. Thus:

$$\vec{A} = \sum_{\lambda} q(t) \vec{A}_{\lambda}(\vec{r}) \tag{1}$$

where $\vec{A}_{\lambda}$ satisfies the equations

$$\nabla^2 \vec{A}_{\lambda} + \frac{\omega_{\lambda}^2}{c^2} \vec{A}_{\lambda} = 0 \tag{2}$$

and

$$\nabla \cdot \vec{A}_{\lambda} = 0 \tag{3}$$

We use the Coulomb gauge throughout this paper. As shown by Heitler the Hamiltonian for the complete system of charged particles and electromagnetic field is

$$H = \sum_{k} \frac{1}{2m_k} \left[\vec{p}_k - \frac{e_k}{c} \sum_{\lambda} q_{\lambda} \vec{A}_{\lambda}(\vec{q}_k) \right]^2 +$$

$$+ \sum_{\lambda} \left(\frac{1}{2} p_{\lambda}^2 + \frac{1}{2} \omega_{\lambda}^2 q_{\lambda}^2\right) + \sum_{i > k} \frac{e_i e_k}{r_{i_k}} \tag{4}$$

We now construct $f^{N, \infty}(q_k, p_K, q_{\lambda}, p_{\lambda})$, the phase-space distribution function of the system. It is a function of all the coordinates and momenta of the N charged particles and of all the coordinates and momenta of the field oscillators. It is the probability density in the infinite dimensional phase

2. W. Heitler, The Quantum Theory of Radiation, 3rd Edition, Oxford: Clarendon Press, 1954, pp. 40-55.

space of the system. The rate of change of $f^{N,\infty}$ is given by Liouville's equation,

$$\frac{\partial f^{N,\infty}}{\partial t} + \left\{ f^{N,\infty},\ H \right\} = 0 \tag{5}$$

It is customary in kinetic theory to speak of functions of position and velocity rather than functions of position and momentum, so before writing this equation in detail we make this change of variables for the particles. Eq. (5) then becomes:

$$\begin{aligned} &\frac{\partial f^{N,\infty}}{\partial t} + \sum_k \vec{v}_k \cdot \frac{\partial f^{N,\infty}}{\partial \vec{q}_k} + \sum_k \frac{\partial f^{N,\infty}}{\partial \vec{v}_k} \\ &\cdot \left[\frac{e_k}{m_k c} \sum_\lambda \left\{ q_\lambda \left[\vec{v}_k \times (\nabla \times \vec{A}_\lambda) \right] - p_\lambda \vec{A}_\lambda \right\} + \right. \\ &\left. - \sum_{i \neq k} \frac{e_k e_i}{m_k} \frac{\partial}{\partial \vec{q}_k} \left(\frac{1}{r_{ik}} \right) \right] + \sum_\lambda p_\lambda \frac{\partial f^{N,\infty}}{\partial q_k} + \\ &+ \sum_\lambda \frac{\partial f^{N,\infty}}{\partial p_\lambda} \left\{ -\omega_\lambda^2 q_\lambda + \sum_k \frac{e_k}{c} \vec{v}_k \cdot \vec{A}_\lambda \right\} = 0 \end{aligned} \tag{6}$$

In the usual way, an equation for $f^k(\vec{q}_k, \vec{v}_k)$, the one-particle distribution function for the k^{th} particle may be found by integrating over all the field oscillators and all the particles except the k^{th}. Likewise an equation may be found for $f^\lambda(q_\lambda, p_\lambda)$, the distribution function for the λ^{th} field oscillator. These are found to be:

$$\begin{aligned} &\frac{\partial f^k}{\partial t} + \vec{v}_k \cdot \frac{\partial f^k}{\partial \vec{q}_k} - \frac{e_k}{m_k c} \sum_\lambda \vec{A}_\lambda(q_k) \cdot \frac{\partial}{\partial \vec{v}_k} \int f^{k,\lambda}\, p_\lambda\, dq_\lambda\, dp_\lambda + \\ &+ \frac{e_k}{m_k c} \sum_\lambda \left\{ \vec{v}_k \times \left[\nabla_k \times \vec{A}_\lambda(\vec{q}_k) \right] \right\} \cdot \frac{\partial}{\partial \vec{v}_k} \int f^{k,\lambda} q_\lambda\, dq_\lambda\, dp_\lambda + \\ &- \sum_{i \neq k} \frac{e_k e_i}{m_k} \int \frac{\partial}{\partial \vec{q}_k} \left(\frac{1}{r_{ik}} \right) \cdot \frac{\partial}{\partial \vec{v}_k} f^{i,k}\, d\vec{q}_i\, d\vec{v}_i = 0 \end{aligned} \tag{7}$$

and

$$\frac{\partial f^{\lambda}}{\partial t} + p_{\lambda} \frac{\partial f^{\lambda}}{\partial q_{\lambda}} - \omega_{\lambda}^{2} q_{\lambda} \frac{\partial f^{\lambda}}{\partial p_{\lambda}} + \sum_{k} \frac{e_k}{c} \int \vec{v}_k \cdot \vec{A}_{\lambda}(\vec{q}_k) \cdot \frac{\partial f^{k,\lambda}}{\partial p_{\lambda}} \, dq_{\lambda} dp_{\lambda} = 0 \qquad (8)$$

It is seen that these equations involve $f^{k,\lambda}$, the pair distribution function for the k^{th} particle and λ^{th} field oscillator, and $f^{i,k}$, the pair distribution function for the i^{th} and k^{th} particles. The Vlasov equation is obtained by assuming that all pair correlations vanish and hence that:

$$f^{k,\lambda}(\vec{q}_k, \vec{v}_k, q_{\lambda}, p_{\lambda}) = f^{k}(\vec{q}_k, \vec{v}_k) \; f^{\lambda}(q_{\lambda}, p_{\lambda}) \qquad (9a)$$

$$f^{i,k}(\vec{q}_i, \vec{v}_i, \vec{q}_k, \vec{v}_k) = f^{i}(\vec{q}_i, \vec{v}_i) \; f^{k}(\vec{q}_k, \vec{v}_k) \qquad (9b)$$

Substituting Eqs. (9) into Eqs. (7) and (8) we find

$$\frac{\partial f^{k}}{\partial t} + \vec{v}_k \cdot \frac{\partial f^{k}}{\partial \vec{q}_k} + \frac{e_k}{m_k c} \frac{\partial f^{k}}{\partial \vec{v}_k} \cdot \Bigg\{ - \sum_{\lambda} \vec{A}_{\lambda}(\vec{q}_k) \int f^{\lambda} p_{\lambda} \, dq_{\lambda} dp_{\lambda} +$$

$$+ \sum_{\lambda} \vec{v}_k \times \left[\nabla_k \times \vec{A}_{\lambda}(\vec{q}_k) \right] \int f^{\lambda} q_{\lambda} \, dq_{\lambda} dp_{\lambda} +$$

$$- \sum_{i \neq k} e_i \frac{\partial}{\partial \vec{q}_k} \int \frac{1}{r_{ik}} f^{i} \, d\vec{q}_i d\vec{v}_i \Bigg\} = 0 \qquad (10)$$

and

$$\frac{\partial f^{\lambda}}{\partial t} + p_{\lambda} \frac{\partial f^{\lambda}}{\partial q_{\lambda}} - \omega_{\lambda}^{2} q_{\lambda} \frac{\partial f^{\lambda}}{\partial p_{\lambda}} + \frac{\partial f^{\lambda}}{\partial p_{\lambda}} \cdot$$

$$\cdot \sum_{k} \frac{e_k}{c} \int \vec{v}_k \cdot \vec{A}_{\lambda}(\vec{q}_k) \; f^{k} d\vec{q}_k d\vec{v}_k = 0 \qquad (11)$$

It is relatively easy to show that Eq. (10) reduces to the Collisionless Boltzmann equation or Vlasov equation:

$$\frac{\partial f^k}{\partial t} + \vec{v}_k \cdot \frac{\partial f^k}{\partial \vec{q}_k} + \frac{e_k}{m_k}\left[\vec{E}_{AV}(\vec{q}_k, t) + \frac{1}{c}\vec{v}_k \times \vec{H}_{AV}(\vec{q}_k, t)\right] \cdot \frac{\partial f^k}{\partial \vec{v}_k} = 0 \qquad (12)$$

The implications of Eq. (11) are examined in the next section. It should be noted that Eq. (11) actually represents a denumerable infinity of equations, one for each oscillator. Likewise Eq. (10) actually represents N equations, one for each particle. In the latter case, the usual assumption of symmetry of the initial ensemble to interchanges between particles of like species reduces Eq. (10) to one equation for each particle species.

III. Entropy of the Radiation Field

Let us define a quantity S_γ such that

$$S_\gamma = -K \sum_\lambda \int f^\lambda(q_\lambda p_\lambda) \ln f^\lambda(q_\lambda p_\lambda)\, dp_\lambda dq_\lambda \tag{13}$$

The time rate of change S_γ is given by

$$\frac{dS_\gamma}{dt} = -K \sum_\lambda \int \left(\frac{\partial f^\lambda}{\partial t} + \frac{\partial f^\lambda}{\partial t} \ln f^\lambda \right) dp_\lambda dq_\lambda \tag{14}$$

The term in $\partial f^\lambda/\partial t$ alone vanishes in virtue of conservation of total probability. (Note that each f^λ and each f^k by virtue of its probability density interpretation must be normalized to unity.) The remaining term can be rewritten by substituting from Eq. (11) and integrating by parts. The result is:

$$\frac{dS_\gamma}{dt} = -K \sum_\lambda \int \Bigg[p_\lambda \frac{\partial f^\lambda}{\partial q_\lambda} - \omega_\lambda^2 q_\lambda \frac{\partial f^\lambda}{\partial p_\lambda} + \frac{\partial f^\lambda}{\partial p_\lambda} \sum_k \frac{e_k}{c} \int \vec{v}_k \cdot \vec{A}_\lambda(\vec{q}_k)\, f^k d\vec{q}_k d\vec{v}_k \Bigg] dq_\lambda dp_\lambda = 0 \tag{15}$$

Thus the entropy of the electromagnetic field is a constant of the motion.

The entropy of the particle field,

$$S = -K \sum_k \int f^k \ln f^k\, d\vec{q}_k d\vec{v}_k \tag{16}$$

is also a constant of the motion. This can be verified immediately by use of Eq. (12) as was shown earlier by Newcomb.[3]

3. I. B. Bernstein; Phys. Rev. 109, 10 (1958), Appendix I.

IV. Entropy Change Due to Incoherence

Suppose that initially all electromagnetic fields are absent. In this case, all the oscillators have completely determined values of their coordinates and canonical momenta, namely zero. The corresponding probability distribution functions of the oscillators is then equal to a product of delta functions:

$$f^{\lambda}(p_{\lambda}\ q_{\lambda}) = \delta(q_{\lambda})\ \delta(p_{\lambda}) \qquad \text{(all } \lambda) \tag{17}$$

If we substitute Eq. (17) in Eq. (13), we find immediately that

$$S_{\gamma} = -\ K \sum_{\lambda} 2\ \ln \delta(0) \tag{18}$$

Suppose now that we excite an electromagnetic field but in a completely coherent fashion. This means that there is not a probability spread of the coordinates and momenta of each oscillator but that instead each oscillator has a precise value for its p and q. In other language, all photons of a given frequency have a unique phase. We expect, in this case, that there has been no disorder created in the oscillator system and that the entropy has remained unchanged. This is indeed the case for we can now write

$$f^{\lambda}(q_{\lambda}\ p_{\lambda}) = \delta(q_{\lambda} - \alpha_{\lambda})\ \delta(p_{\lambda} - \beta_{\lambda})$$

where α_{λ} and β_{λ} are the precise values of the λ^{th} oscillator's coordinate and momentum. Upon substitution in Eq. (13) this gives precisely the same result as in Eq. (18).

Now suppose that there is some incoherence. A trivial example will demonstrate that the entropy in the field will increase. Suppose that one oscillator, say the j^{th}, has a probability spread of its momenta. Say it has a $p_j = \beta_j$ with probability 1/2 and a $p_j = \gamma_j$ with equal probability. Now

$$f^j(q_j p_j) = \delta(q_j - \alpha_j)\left[\frac{1}{2}\,\delta(p_j - \beta_j) + \frac{1}{2}\,\delta(p_j - \gamma_j)\right]$$

while for all the other oscillators,

$$f^\lambda(q_\lambda p_\lambda) = \delta(q_\lambda - \alpha_\lambda)\,\delta(p_\lambda - \beta_\lambda) \qquad (\lambda \neq j)$$

Upon substitution of these equations in Eq. (13) we find

$$\begin{aligned} S = & - K \sum_{\lambda \neq j} 2\,\ln \delta(0) + \\ & - K \int \delta(q_j - \alpha_j)\left[\frac{1}{2}\,\delta(p_j - \beta_j) + \frac{1}{2}\,\delta(p_j - \alpha_j)\right]\cdot\Big\{\ln \delta(q_j - \alpha_j) + \\ & + \ln\left[\frac{1}{2}\,\delta(p_j - \beta_j) + \frac{1}{2}\,\delta(p_j - \gamma_j)\right]\Big\}\, dq_j dp_j \\ & = - K \sum_{\lambda \neq j} 2\ln \delta(0) - K \ln \delta(0) - K \ln\left[\frac{1}{2}\,\delta(0)\right] \\ & = - K \sum_{\lambda} 2\ln \delta(0) + K \ln 2 \end{aligned}$$

Hence the entropy has increased over that for a completely coherent radiation field.

These considerations go through with some trivial extensions if static external fields are present originally.

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