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TOPICS ON THE NUMERICAL SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS

R. C. F. Bartels

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MATHEMATICS PANEL

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DATE ISSUED

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TOPICS ON THE NUMERICAL SOLUTION
OF
PARTIAL DIFFERENTIAL EQUATIONS

R. C. F. Bartels

I. Introduction

§1 Properly posed problems in partial differential equations

Our discussion deals with the numerical solution of partial differential equations. For the purpose of illustrating the methods and concepts that are studied, we will consider examples involving the well known classical equations*:

$$u_{xx} + u_{yy} = 0 \quad ; \tag{1.1}$$

$$u_t = u_{xx} \quad ; \tag{1.2}$$

$$u_{tt} = u_{xx} \quad . \tag{1.3}$$

These are prototypes of more general partial differential equations of the second order that are important in the physical sciences. We will not limit the applicability of the ideas which are introduced to equations in two independent variables, nor to equations that are necessarily linear. We will also deal with a system involving more than one dependent variable.

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*For a relatively complete discussion of these partial differential equations see, for example, the reference [1] or [2] listed in the bibliography.

The type of problem which we will consider is that in which the solution satisfies the appropriate partial differential equation together with certain prescribed supplementary conditions. These supplementary conditions are usually:

(a) conditions prescribed on the boundary of the region of space in which the solution of the partial differential equation is to be found, i.e., the so-called boundary conditions;

(b) (if time is involved) conditions prescribed at some definite value of the time; the so-called initial conditions.

The differential equation, together with the supplementary conditions, is called a boundary-value or an initial value problem. We will consider only those problems which, in the terminology of Hadamard*, are properly posed; i.e., a problem for which its solution exists, is unique, and depends continuously on the prescribed supplementary conditions.

It is important to remark that it is not always possible to assign arbitrarily the two types of supplementary conditions (a) and (b) to a given differential equation in the formulation of a properly posed problem. In fact, partial differential equations partion themselves in mutually exclusive classes depending in some sense on the type of supplementary conditions with which they will constitute a properly posed problem. The classification of the linear, second order equations in two independent variables, such as those in (1.1), (1.2), (1.3), can be expressed very simply in terms of the coefficients of the highest ordered derivatives.

---

*See Hadamard [3] .

More generally, an equation of the form

$$A(x,y)\, u_{xx} + 2B(x,y)u_{xy} + C(x,y)u_{yy} + D(x, y, u, u_x, u_y) = 0 ,$$

where the coefficients A , B , and C depend only on x and y , is called elliptic, parabolic, or hyperbolic in a domain of the (x, y)-plane according as the determinant

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix}$$

is positive, negative, or zero throughout the domain. According to this criteria the equations (1.1), (1.2), and (1.3) are classified in the order just named. The type of supplementary condition that is appropriate for each of these types is illustrated in the subsequent examples.

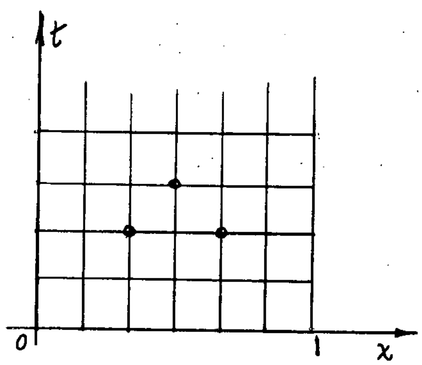
§2 Finite difference approximations of partial differential equations

The usual method of obtaining an approximate solution of a properly posed problem for a partial differential equation is to replace the derivatives in the equation and supplementary conditions by finite differences. In this procedure, the space variables assume the discrete values corresponding to the points of a mesh or lattice imbedded in the space and "filling" the region. The time variable, when it appears, also assumes discrete values $t_0, t_1, \ldots, t_n, \ldots,$ where $t_n = n \Delta t$.

Accordingly, the solution u is approximated at the discrete points of space and time.

For example, consider the simple initial value problem for the parabolic equation (1.1) in which the function $u(x, t)$ is determined by the conditions

$$(2.1)\quad \begin{cases} u_t = u_{xx}, & (0 < x < 1, t > 0), \\ u(0, t) = \varphi(t), \quad u(1, t) = \psi(t), & (t > 0), \\ u(x, 0) = f(x) & (0 < x < 1). \end{cases}$$



Let the continuous variables x and t be replaced by the discrete set

$$(x_m, t_n), \quad (m = 0, 1, \ldots, M), \quad (n = 0, 1, \ldots),$$

where

$$x_m = m \Delta x, \quad t_n = n \Delta t, \quad \text{and} \quad \Delta x = 1/M.$$

Then the simplest finite difference approximation to the problem (2.1) is

$$(2.2)\quad \begin{cases} \dfrac{v(x_m,\ t_n+\Delta t) - v(x_m,t_n)}{\Delta t} = \dfrac{v(x_m+\Delta x,t_n) - 2v(x_m,t_n) + v(x_m-\Delta x,t_n)}{(\Delta x)^2} \\ \qquad\qquad (m = 1,\ 2,\ \ldots,\ M-1, \quad n \geqslant 0), \\ v(0,\ t_n) = \varphi(t_n), \quad v(1,\ t_n) = \psi(t_n) \quad , \qquad (n \geqslant 0) ; \\ v(x_m,\ 0) = f(x_m) \quad , \qquad (m = 1,\ 2,\ \ldots,\ M-1) \quad . \end{cases}$$

In this case, the difference equation is obtained by replacing the time derivative by a forward difference, and the space derivative by the central difference on the line $t = t_n$. The difference equation in (2.2) can also be written in the form

$$(2.3)\quad v(x_m,t_{n+1}) = \lambda\, v(x_{m+1},\ t_n) + (1-2\lambda)\, v(x_m,t_n) + \lambda\, v(x_{m-1},\ t_n) \quad ,$$

where

$$(2.4)\qquad \lambda = \Delta t/(\Delta x)^2 \quad .$$

In this form, it is clear that the values of v at the "interior" points of the mesh contained within the strip $0 < x < 1$, $t > 0$ can be calculated successively, line by line, starting from the initial values

$v(x_m, 0) = f(x_m)$ and making use of the boundary data at each step to evaluate $v(x_m, t_n)$ for $m = 0$ and M. This is the so-called, explicit scheme for approximating the solution of (2.1).

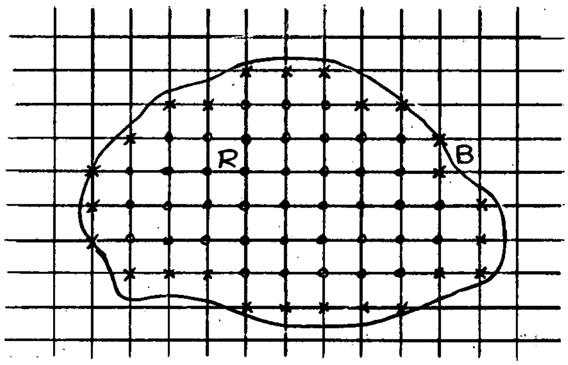
On the other hand, an alternate finite difference approximation of the problem (2.1) can be obtained by replacing the second order space derivative by a central difference on the line $t = t_{n+1}$. In this way, one obtains the so-called implicit scheme for approximating the solution of (2.1), namely,

$$(2.5)\quad \begin{cases} \lambda\, v(x_{m+1},t_{n+1}) - (1+2\lambda)\, v(x_m,t_{n+1}) + \lambda\, v(x_{m-1},t_{m+1}) = -v(x_m,t_n) \\ \qquad (m = 1, 2, \ldots, M-1;\ n \geqslant 0), \\ v(0, t_n) = \varphi(t_n) \;,\quad v(1, t_n) = \psi(t_n) \;,\quad (n > 0)\;, \\ v(x_m, 0) = f(x_m) \;. \end{cases}$$

In this case the difference equation does not express the values of v on the line $t = t_{n+1}$ explicitly in terms of its values along the line $t = t_n$. However, the difference equations in (2.5), for $m = 1, 2, \ldots, M-1$, constitute a system of $(M-1)$ algebraic equations for the unknown values of v at the $(M-1)$ interior grid points on the line $t = t_{n+1}$.

$$(x_m, t_n) \;,\qquad m = 1, 2, 3, \ldots, (M-1) \;.$$

As a second example, let us consider the Dirichlet problem for Laplace's equation in the two dimensional domain R with boundary B :



$$(2.6)\qquad \begin{cases} u_{xx} + u_{yy} = 0 \ , & (x, y) \text{ in } R \ , \\ u = f(x, y) \ , & (x, y) \text{ on } B \ . \end{cases}$$

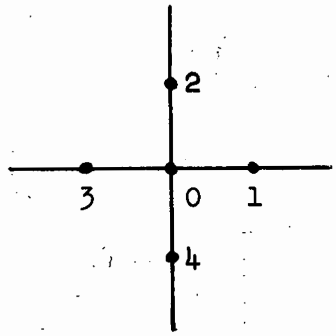
In order to approximate the solution of this problem by a finite difference procedure, let us replace the continuous variables x and y by the discrete set

$$(x_m, y_n) \equiv (\bar{x} + mh \ , \bar{y} + nh) \quad , \quad m, n = 0, 1, 2, \ldots,$$

for some arbitrary $(\bar{x}, \bar{y})$ and for $h > 0$. If $z_0 \equiv (x_0, y_0)$ is a mesh point, then the four points

$$z_1 \equiv (x_0 + h, y_0) \quad , \quad z_2 = (x_0, y_0 + h) \ , \quad z_3 = (x_0 - h, y_0)$$

$$z_4 = (x_0, y_0 - h)$$

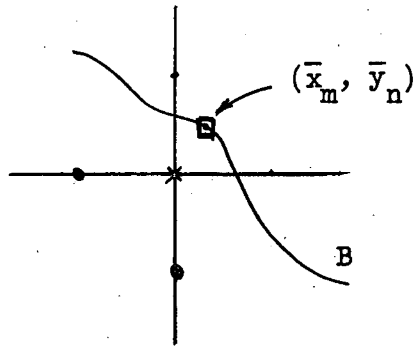


are called the <u>neighbors</u> of z_0 . This set of four points will be denoted by $N(z_0)$. We shall also introduce the notation:

$$
(2.7)\quad \begin{cases} R_h & \text{is the set of lattice points } z \text{ such that the set of} \\ & \text{neighbors } N(z) \text{ belongs to the closed region } R + B \;; \\ & \\ B_h & \text{is the set of lattice points belonging to } R + B \text{ but} \\ & \underline{\text{not}} \text{ to } R_h \;. \end{cases}
$$

A simple difference approximation of the problem (2.6) is then

$$
(2.8)\quad \begin{cases} v(x_m, y_n) = \frac{1}{4}\left\{v(x_{m+1}, y_n) + v(x_m, y_{n+1}) + v(x_{m-1}, y_n) + v(x_m, y_{n-1})\right\}, \\ \qquad\qquad\qquad\qquad\qquad\qquad (x_m, y_n) \text{ in } R_h \\ \\ v(x_m, y_n) = f_h \,, \qquad (x_m, y_n) \text{ in } B_h \,, \end{cases}
$$



where $f_h = f(\bar{x}_m, \bar{y}_n)$, and $(\bar{x}_m, \bar{y}_n)$ is any point on the boundary B such that

$$(x_m - \bar{x}_m)^2 + (y_n - \bar{y}_n)^2 \leqslant h^2 . \tag{2.9}$$

More simply, this formulation of the difference problem constitutes a system of, say, M linear algebraic equations which express the values of the approximate solution v at the M interior lattice points R_h in terms of the prescribed values on the boundary points B_h .

There are many more examples that we might give at this point. However, these will serve to motivate the immediate discussion.

At this point, let us observe that if the function $w(x, y)$, for example (the fact that we have chosen the independent variables x and y is not significant), together with its partial derivatives w_x , w_{xx} , and w_y , are continuous in a closed region of the (x, y)-plane, then at any interior point (x, y) of this region

$$\lim_{\Delta x \to 0} \left\{ \frac{w(x+\Delta x, y) - 2w(x,y) + w(x-\Delta x, y)}{(\Delta x)^2} - w_{xx}(x, y) \right\}$$

$$= \lim_{\Delta x \to 0} \left\{ w_{xx}(x + \theta_1 \Delta x, y) - w_{xx}(x,y) \right\} = 0 \quad ,$$

and

$$\lim_{\Delta y \to 0} \left\{ \frac{w(x, y+\Delta y) - w(x,y)}{\Delta y} - w_y(x,y) \right\}$$

$$= \lim_{\Delta y} \left\{ w_y(x, y + \theta_2 \Delta y) - w_y(x,y) \right\} = 0 \quad ,$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$. These relations are easily derived with the aid of Taylor's formula with remainder. It therefore follows that if, on the one hand, the functions $v(x, t)$ in equations (2.2) and (2.5) is sufficiently smooth in the closed region $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$, and, on the other, the function $v(x, y)$ in (2.8) is sufficiently smooth in the closed region $R + B$, then the finite difference quotients in the equations (2.2), (2.5), and (2.8) will converge to the corresponding derivatives in the differential equations (2.1) and (2.6), respectively, as the increments Δx, Δt, and Δy tend to zero. Consequently, the finite difference problems in (2.2), (2.5), and (2.8) are said to be <u>consistent approximations</u> of the corresponding initial or boundary value problem.

In the sequel we will consider only finite difference approximations of initial or boundary value problems which satisfy a criterion of consistency similar to that described above. It does not, however, follow that this criterion is always sufficient to insure that the solution of the finite difference problem will converge to the solution of the approximated differential problem as the mesh size (i.e., the increments in the variables) tends to zero. This remark will be elaborated upon in the next paragraph.

§3 Truncation errors and the problem of the convergence of difference approximations.

Let v denote the exact solution of a finite difference approximation of a given initial or boundary value problem for a partial differential equation. Strictly speaking, v belongs to an infinite sequence of approximate solutions corresponding to an increasingly finer mesh. If u denotes the true solution of the boundary value problem, then the basic question is whether the sequence of difference approximations v converge to the solution u as the mesh size tends to zero. The difference between these two quantities, namely,

$$w = u - v \tag{3.1}$$

is called the truncation error. The problem of the convergence of the finite difference approximation is therefore the problem of showing that the corresponding truncation error w tends to zero with the mesh.

The problem of convergence is not as important to the person performing an actual computation as the problem of obtaining an explicit appraisal of the magnitude of the truncation error at any step of the calculation. In general this problem is a more difficult one. In the subsequent sections we will indicate by illustrations a method for establishing the convergence of a wide class of finite difference approximations which will, at the same time, yield some information on the magnitude of the truncation error.

§ 4 Computational stability of finite difference approximations*

Because of the limitations of the computing machine, it is generally not possible to obtain the exact solution v of the finite difference problem. The machine performs arithmetic operations in terms of quantities which are "rounded-off" approximations of the exact values. Consequently, instead of obtaining the exact solution v of the finite difference problem coded for the machine, an approximate machine calculation v^* is obtained. The difference

$$s = v^* - v \tag{4.1}$$

represents the remainder or departure of the approximate solution v^* from the true solution v. The departure s is an accumulation of the error

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*The ideas expressed here were suggested by the lectures of W. Wasow on the same subject. It is understood that a very extensive treatment of the stability of partial differential equations by Wasow will soon make its appearance in book form.

introduced at each step of the calculation, i.e., at each mesh point. The ideal behavior of a finite difference approximation is that in which the maximum numerical value of the departure s in the given domain of the variables tends to zero uniformly with respect to mesh size as the magnitude of the individual errors at each step tends to zero. Thus, if the mesh size depends on a single parameter h and the maximum numerical value of the errors introduced at each step is estimated as δ, then the ideal situation would be that $\max|s| \longrightarrow 0$ as $\delta \longrightarrow 0$ uniformly with respect to h. However, this ideal <u>can not</u> be expected in general for even the case of linear finite difference problems.

Since there always exists bounds for the rate of growth of the magnitude of s as a function of the mesh size parameter h for a fixed region of the variables, the maximum departure can in principle be controlled. In other words, by carrying out the calculation with sufficient precision, the computed quantity v^* can be made to agree as closely as may be desired with the exact solution v of the finite difference problem. However, the precision necessary in a given situation might exceed the capabilities of any computing machine. A finite difference method for which the latter is experienced might for the purpose of classification be termed unstable.

To be more precise, we will consider the stability of a finite difference procedure as a relative property of the procedure which will be measured in some sense by the order of magnitude of the maximum departure s. For example, if the maximum numerical value of the departure s in a fixed region is proportional to δh^{α} for some real (but not necessarily positive) number α as the magnitude, δ , of the errors introduced at each step and the mesh size, h , tend to zero, then the finite difference procedure will be considered stable; α is an index of the degree of relative stability. A finite difference procedure in which the magnitude of the departure s is proportional to $\exp(h^{-1})$, say, as $h \longrightarrow 0$ will, on the other hand, be called unstable.

Unfortunately, the precise order of magnitude of the maximum departure s as a function of δ and h is not easily determined in general. Several cases in which an estimate of the departure can be effected will be given in the subsequent sections.

II. ELLIPTIC DIFFERENTIAL EQUATIONS

§5 Truncation errors in the solution of Laplace's equation by finite differences.

Let us now deal with the problem of estimating the truncation error which results when the Dirichlet problem (2.6) for Laplace's equation is replaced by the finite difference problem formulated in (2.8). That is, we seek to obtain some appraisal for the maximum numerical value of the difference $w = u-v$, where u is the solution of the Dirichlet problem (2.6) and v is the solution of the finite difference problem (2.8).

For the purposes of convenience, let us introduce the notation

$$(5.1) \qquad L_h\left[v\right] \equiv \frac{1}{4h^2}\left\{v(x+h,y) + v(x,y+h) + v(x-h,y) + v(x,y-h) - 4v(x,y)\right\}$$

$$\equiv \frac{1}{h^2}\left\{\sum_{i=1}^{4} \frac{1}{4} v(x_i, y_i) - v(x, y)\right\} ,$$

where $v(x_i, y_i)$, $(i = 1, 2, 3, 4)$, are the four neighbors of (x, y). Then a solution v of the finite difference problem (2.8) satisfies the equation

$$(5.2) \qquad L_h\left[v\right] = 0 \quad \text{for } (x, y) \text{ in } R_h .$$

On the other hand, if the harmonic function $u(x, y)$ satisfying the conditions of the Dirichlet problem is continuous and its partial derivatives up to and including those of the fourth order exist and are bounded in the closed region $R + B$, then, at every interior point of R , it is true that

$$(5.3) \qquad L_h\left[u\right] = \frac{h^2}{96}\left\{u_{xxxx}(x+\theta_1 h,y) + u_{xxxx}(x,y+\theta_2 h) + u_{xxxx}(x-\theta_3 h,y) + u_{xxxx}(x,y-\theta_4 h)\right\} ,$$

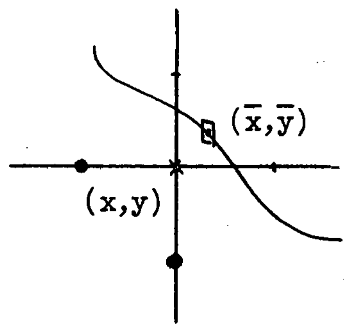
where θ_1 , θ_2 , θ_3 , θ_4 are numbers between 0 and 1 . More simply,

$$(5.4) \qquad L_h\left[u\right] = \rho(x,y) \quad \text{for } (x, y) \text{ in } R_h ,$$

where $\rho(x, y)$ denotes the right-hand member of (5.3). Consequently, in accordance with (5.2) and (5.4), the truncation error w satisfies the relation

$$(5.5) \qquad L_h\left[w\right] = L_h\left[u\right] - L_h\left[v\right] = L_h\left[u\right] = \rho(x,y)$$

for all points in R_h .



If (x, y) is a point of the boundary set B_h , then it follows from (2.6) and (2.8) that

$$w(x, y) = u(x, y) - v(x, y)$$

$$= u(x, y) - u(\bar{x}, \bar{y})$$

where $(\bar{x}, \bar{y})$ is a point of B such that

$$(x - \bar{x})^2 + (y - \bar{y})^2 \leqslant h^2 .$$

Therefore,

$$w(x, y) = \sigma(x, y) \quad \text{for} \quad (x, y) \quad \text{in} \quad B_h , \tag{5.6}$$

where

$$\sigma(x, y) = u_x(x',y')\,(x - \bar{x}) + u_y(x',y')\,(x - \bar{x}) , \tag{5.7}$$

and (x',y') is a point lying between (x, y) in B_h and $(\bar{x},\bar{y})$ on B .

We have thus shown that the truncation error w is a solution of a finite difference problem on the given mesh. It is of course true that the non-homogeneous terms $\rho(x, y)$ and $\sigma(x, y)$ which appear in the difference equation (5.5) and boundary condition (5.6) depend on a knowledge of the solution u of the difference problem. We are not, however, interested in the precise value of the truncation error at each point of the mesh. We shall show that in order to obtain an appraisal of the maximum numerical value of this error it is sufficient to know the bounds on the partial derivatives of u in the closed region $R + B$. This appraisal is based on the so called maximum principal for a finite difference equation of positive type*. For this purpose we need two lemmas.**

<u>Lemma 1</u>. If $L_h\left[w\right] \geqslant \delta > 0$ for all points of R_h, then

(5.8) $$\max_{R_h} w < \max_{B_h} w \quad .$$

Proof: It is clear from (5.1) that the inequality $L_h\left[w\right] \geqslant \delta$ implies that for every point (x_0, y_0) of R_h

(5.9) $$w(x_0,y_0) \leqslant \sum_{i=1}^{4} \frac{1}{4}\, w(x_i, y_i) - h^2 \delta \quad ,$$

where (x_i, y_i) $(i = 1, 2, 3, 4)$ are the neighbors of (x_0, y_0). Now suppose that the maximum value of w does not occur at the boundary points B_h but, instead, at some point of R_h, so that

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*See the general treatment of equations of this type by Motzkin and Wasow [4].

**See Wasow [5] for generalizations of these lemmas and results.

$$w \leqslant \max_{R_h} w$$

for every point of $R_h + B_h$. Then it would follow from (5.9), since the coefficients are positive and have sum equal to 1 , that

$$w(x_0, y_0) \leqslant \max_{R_h} w - h^2 \delta$$

for <u>every</u> point (x_0, y_0) in R_h . This immediately leads to a contradiction on taking (x_0, y_0) a point of R_h at which w assumes its maximum value.

<u>Lemma 2</u>. If $\left| L_h \left[w \right] \right| \leqslant k$ for all points of R_h , then

$$\max_{R_h} |w| \leqslant k\, d^2 + \max_{B_h} |w| \quad , \tag{5.9}$$

where k is a non-negative constant and d is the "diameter" of the region R .

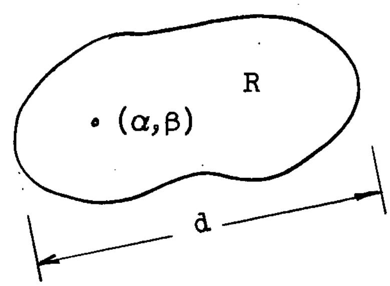
Proof: We first define the function $q(x, y)$ such that

$$q(x, y) \geqslant 0 \quad \text{and} \quad L_h \left[q \right] = +1 \, , \quad \text{for} \quad (x, y) \quad \text{in} \quad R_h + B_h \, .$$

For example, these conditions are satisfied if

$$q(x, y) = (x - \alpha)^2 + (y - \beta)^2$$

for any point (α, β) in $R + B$.



Now let $k^* > k$. Then, for all points of R_h,

$$L_h\left[k^* q \pm w\right] = k^* \pm L_h\left[w\right] > 0 .$$

Hence, applying the result of lemma 1 to the function $k^* q \pm w$, we have

$$\max_{R_h} (k^* q \pm w) \leqslant \max_{B_h} (k^* q \pm w)$$

(5.10) $$\leqslant k^* \max_{B_h} q + \max_{B_h} (\pm w)$$

$$\leqslant k^* d^2 + \max_{B_h} |w| ,$$

where d is the diameter of R. Also, since $k^* q \geqslant 0$ in R_h,

(5.11) $$\max_{R_h} |w| = \max_{R_h} (w_1 - w) \leqslant \max_{R_h} (k^* q \pm w) .$$

Hence, by combining (5.10) and (5.11), it follows that, for all $k^* > k$,

$$\max_{R_h} |w| \leqslant k^* d^2 + \max_{B_h} |w| .$$

The inequality (5.9) immediately follows on letting $k^* \longrightarrow k$.

We now make the following appraisal of the truncation error for the finite difference problem (2.8).*

<u>Theorem</u>. Let u and v be solutions of the problems (2.6) and (2.8), respectively. If u and its derivatives up to and including those of the fourth order exist and are bounded in the closed region $R + B$, then the truncation error $w = u - v$ is such that

$$\max |w| \leqslant \bar{\rho}\, d^2 h^2 + \bar{\sigma}\, h \qquad (5.12)$$

where $\bar{\sigma}$ and $\bar{\rho}$ depend on the bounds for the numerical values of the first and fourth partial derivatives, respectively, in $R + B$.

The proof of this theorem follows immediately on applying the result of lemma 2 to the particular finite difference problem formed by (5.4) and (5.6) of which the truncation error is a solution.

§6 Generalizations and improvements of the previous results.

It is possible to extend the results of the previous paragraph to boundary value problems with more general differential equations of the elliptic type. For example, consider the difference operator

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*The result stated here was first obtained by Gerschgorn in [6].

$$(6.1) \qquad L_h\left[w\right] \equiv \frac{1}{h^2}\left\{c_1w(z_1) + c_2w(z_2) + c_3w(z_3) + c_4w(z_4) - c_0w(z_0)\right\}$$

where z_1 , z_2 , z_3 , z_4 are neighbors of z_0 , the coefficients c_1 , c_2 , c_3 , c_4 are non-negative, and $c_0 > 0$ in $R_h + B_h$. It is easily seen that the conclusions of lemma 1 apply for this difference operator provided that*

$$(6.2) \qquad c_0 \geqslant c_1 + c_2 + c_3 + c_4 \quad \text{in } R_h + B_h \ .$$

The details of the proof of the lemma for this more general case are unaltered. The extension of lemma 2 then follows without difficulty.

In particular, suppose that the differential equation in the boundary value problem (2.6) is replaced by the equation of the elliptic type

$$(6.3) \qquad A\frac{\partial^2 u}{\partial x^2} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \ ,$$

where the coefficients A , ..., G and their second order derivatives are continuous, $A > 0$, $C > 0$, and $F \leqslant 0$ in the domain R . This equation can be approximated by the difference equation

$$(6.4) \qquad L_h\left[v\right] = G \ ,$$

*See Motzkin and Wasow [4], p. 257.

with the difference operator defined by (6.1) in which

$$c_0 = 2(A + C - \frac{1}{2} Fh^2) \quad ,$$

$$c_1 = (A + \frac{1}{2} Dh) \; , \quad c_2 = (C + \frac{1}{2} Eh) \; , \quad c_3 = (A - \frac{1}{2} Dh) \; , \quad c_4 = (C - \frac{1}{2} Eh) \; .$$

If the increment h is then chosen so that

$$\frac{h}{2} < \min \left\{ \frac{\min A}{\max |D|} \; , \; \frac{\min C}{\max |E|} \right\} \; , \tag{6.5}$$

the coefficients c_0 , c_1 , c_2 , c_3 , c_4 satisfy the conditions in (6.1) and (6.2). The foregoing extensions of the results of the last section then yields the result:*

<u>Theorem</u>. Let u be the solution of the Dirichlet problem for the elliptic equation (6.3), and let v be the solution of the corresponding finite difference problem with the equation (6.4). If h satisfies (6.5), and if u and its derivatives up to and including those of fourth order exist and are bounded in the closed region $R + B$, then the truncation error $w = u - v$ is such that

$$\max |w| \leqslant M_1 h + M_2 h^2 \quad , \tag{6.6}$$

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*See Gerschgoren [6] for details of this extension.

where M_1 is proportional to the maximum numerical value of the first partial derivatives of u at the boundary B and M_2 is an upper bound depending on the coefficients of the differential equation and the derivatives of u including those of the fourth order.

Motzkin and Wasow* have considered general differential expressions of the form

$$\sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_k} + \sum_{j=1}^{n} b_j(x) \frac{\partial u}{\partial x_j} + c(x)u \quad ,$$

where x is a point with coordinates x_1 , x_2 , ..., x_n , which are uniformly elliptic in the closed region $R + B$, that is,

$$\det \left\{ a_{ik} (x) \right\} \geqslant \text{const.} > 0 \qquad \text{in } R + B \, .$$

They have shown that for sufficiently small h there always exists a difference expression which is a consistent approximation of the differential expression and which satisfies a maximum principle analogous to that of our lemma 1.

Our previous results can be improved in another direction. Collatz has shown** that by a more careful choice of the values at the boundary points B_h , the truncation error w satisfies the relation

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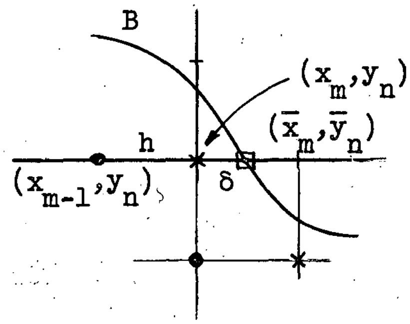
*See [4].

**For details, see Collatz [7].

$$\max_{R_h} |w| \leqslant M_1' h^2 + M_2 h^2 \quad ,$$



where M_1' and M_2 are constants depending on derivatives of the solution of the partial differential equation. For example, this appraisal of the truncation error results if the values at the boundary point (x_m, y_n) are defined by the weighted average (see figure):

$$v(x_m, y_n) = \frac{h\, v(x_{m-1}, y_n) + \delta\, f(\bar{x}_m, y_n)}{h + \delta} \quad .$$

This is equivalent to replacing the set of boundary values in (2.10) by the set of non-homogeneous equations

$$(h + \delta)\, v(x_m, y_n) - h\, v(x_{m-1}, y_n) = \delta\, f(\bar{x}_m, y_n) \quad .$$

All of the results which we have stated above suffer from the defect that the appraisals obtained depend on the bounds for the derivatives of the unknown solution of the differential equation itself. These bounds cannot be found in the general case without effectively solving the differential equation. In the case of Laplace's equation, strict estimates of these bounds in terms of the boundary data can be obtained at interior points of the mesh, provided that the boundary B and the boundary data are sufficiently smooth.* Indeed, Wasow has resently extended an estimate of these bounds to the case in which the function prescribed on the boundary are piecewise continuous.**

§7 Stability of the finite difference approximation of the Dirichlet problem

The finite difference problem in (2.8) is inherently stable. This is easily demonstrated with the results of the foregoing sections. As in §4, let the machine calculations of the solution of the difference problem be denoted by v^*. Then, since the computed values do not in general satisfy the difference equation (5.2), or the boundary condition, we have

$$(7.1) \qquad L_h\left[v^*\right] = \delta_1 h^{-2} \qquad \text{for } (x, y) \text{ in } R_h ,$$

and

$$(7.2) \qquad v^* = f_h + \delta_2 \qquad \text{for } (x, y) \text{ on } B_h ,$$

where $\delta_1 h^{-2}$ and δ_2 represent the errors stemming from the rounding-off of the arithmetic operations. Subtracting these equations from the corresponding equations in (2.8) and using the notation of (4.1), one

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*See for example Rosenbloom [8].

**This work by Wasow is awaiting publication.

obtains the following relations for the departure $s = v^* - v$:

(7.3) $$L_h[s] = \delta_1 h^{-2} \qquad \text{for } (x, y) \text{ in } R_h ,$$

(7.4) $$s = \delta_2 \qquad \text{for } (x, y) \text{ on } B_h .$$

Hence, in accordance with lemma 2,

$$\max_{R_h} |s| \leqslant \delta (M_1 + M_2 h^{-2})$$

where M_1 and M_2 are constants and

$$\delta = \max_{R_h+B_h} (|\delta_1| , |\delta_2|) .$$

Therefore

$$s = 0 (\delta h^{-2}) \qquad \text{as} \qquad h \longrightarrow 0 ,$$

whence the procedure is stable.

III. PARABOLIC DIFFERENTIAL EQUATIONS

§8 Stability of the simple finite difference problem of parabolic type.

Let us return to the simple finite difference problem for the parabolic partial differential equation that is formulated in equations (2.2), (2.3), and (2.4). If we set $\Delta x = h$, $\Delta t = k = \lambda h^2$, where the ratio

$$\lambda = \Delta t/(\Delta x)^2$$

is regarded as fixed, then the difference equation in (2.3) can be expressed in the form

$$(8.1) \qquad v(x, t+k) = \sum_{r=-1}^{+1} c_r \, v(x + rh, t)$$

where

$$(8.2) \qquad c_{-1} = \lambda \ , \quad c_0 = (1 - 2\lambda) \ , \quad c_1 = \lambda \ , \quad \text{and} \ \sum_r c_r = 1 \ .$$

Again, for simplicity in writing, we use the notation

$$(8.3) \qquad L_h \left[v \right] \equiv \frac{1}{k} \left\{ v(x, t+k) - \sum_r c_r \, v(x + rh, t) \right\} \ .$$

Let us first observe that for $0 < \lambda \leqslant \frac{1}{2}$ the right hand member of equation (8.1) is a weight average of v at three neighboring mesh points with non-negative weight factors. Therefore the value of v at any interior mesh point $(x, t+k)$ will lie between the upper and lower bounds of its values at the mesh points on the line t ; that is, for $0 < x < 1$,

$$\min_{0<x<1} v(x, t) \leqslant v(x, t+k) \leqslant \max_{0<x<1} v(x,t) \quad . \tag{8.4}$$

We now establish a result which is analogous to the principal of maximum obtained in lemma 2 for elliptic equations.

<u>Theorem</u>. Let $v(x, t)$ be a solution of the difference problem

$$\begin{cases} L_h\left[v\right] = \rho(x, t) , & (0 < x < 1 , t > 0) \\ v(0, t) = \varphi(t) , \quad v(1, t) = \psi(t) , & (t > 0) \\ v(x, 0) = f(x) , & (0 < x < 1) . \end{cases} \tag{8.5}$$

If $0 < \lambda \leqslant \frac{1}{2}$, then

$$|v(x, t)| \leqslant \bar{\rho}t + \max(\bar{f}, \bar{\varphi}, \bar{\psi}) , \; (0 \leqslant x \leqslant 1 , \; 0 \leqslant t \leqslant T) , \tag{8.6}$$

where $\bar{f}, \bar{\varphi}, \bar{\psi}, \bar{\rho}$ are the least upper bounds of the functions $|f|$, $|\varphi|$, $|\psi|$, $|\rho|$, respectively, in $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$.

Proof: Let us first observe that we can reduce our discussion to difference problems in which the difference equation is homogeneous. To begin with, let $V_\alpha(x, t)$ be defined by the conditions

(8.7a) $$L_h\left[V_\alpha\right] = 0 \quad \text{for} \quad t \geqslant (\alpha+1)k \,, \; 0 < x < 1$$

(8.7b) $$L_h\left[V_\alpha\right] = \rho(x,t) \quad \text{for} \quad t = \alpha k \,, \quad 0 < x < 1$$

(8.7c) $$V_\alpha(x, t) = 0 \quad \text{for} \quad t \leqslant \alpha k \,, \quad 0 < x < 1$$

(8.7d) $$V_\alpha(0, t) = V_\alpha(1, t) = 0 \,, \quad t > 0 \;.$$

Then the function

(8.8) $$v_1(x, t) = \sum_{\alpha=0}^{t/k} V_\alpha(x, t)$$

is a particular solution of the difference equation in (8.5) that vanishes for $t = 0$, $x = 0$, and $x = 1$. But, from (8.7b) and (8.7c), it follows that when $t = \alpha k$,

$$V_\alpha(x, t+k) = \sum_r c_r \, V_\alpha(x+rh,t) + k\rho(x,t) = k\,\rho(x, t) \;.$$

Therefore the function $V_\alpha(x, t)$ is a solution of the homogeneous initial value problem

$$(8.9)\qquad \begin{cases} L_h\left[V_\alpha\right] = 0\ , & t > (\alpha+1)k\ ,\quad 0 < x < 1 \\ V_\alpha(x,t) = k\,\rho(x,t)\ , & t = (\alpha+1)k\ ,\quad 0 < x < 1 \\ V_\alpha(x,t) = 0\ , & t \leqslant \alpha k\ ,\quad 0 < x < 1 \\ V_\alpha(0,t) = V_\alpha(1,t) = 0\ , & t > 0\ . \end{cases}$$

Hence, we can express the solution of the difference problem (8.5) in the form

$$v(x,t) = v_0(x,t) + v_1(x,t) = v_0(x,t) + \sum_{\alpha=0}^{t/k} V_\alpha(x,t)\ ,$$

where $v_0(x,t)$ is also a solution of a problem with a homogeneous difference equation, namely,

$$(8.10)\qquad \begin{cases} L_h\left[v_0(x,t)\right] = 0\ , & (0 < x < 1\ ,\ t > 0) \\ v_0(0,t) = \varphi(t)\ ,\ v_0(1,t) = \psi(t)\ , & (t > 0) \\ v_0(x,0) = f(x)\ , & (0 < x < 1). \end{cases}$$

Consider the function $v_0(x, t)$. It follows from (8.4) that $|v_0(x, t)| \leqslant \bar{F}$ at interior mesh points of the line $t = k$. Therefore

$$|v_0(x,k)| \leqslant \max(\bar{f}, \bar{\varphi}, \bar{\psi}) \quad \text{for} \quad 0 \leqslant x \leqslant 1 .$$

The same argument can be repeated for the step from $t = k$ to $t = 2k$, etc. In general, we obtain

$$|v_0(x,t)| \leqslant \max(\bar{f}, \bar{\varphi}, \bar{\psi}) \quad \text{for} \quad 0 \leqslant x \leqslant 1 , \quad 0 \leqslant t \leqslant T . \tag{8.11}$$

The numerical values of the functions V_α in (8.9) can be estimated in the same way. In fact, since these functions vanish for $x = 0$ and $x = 1$, we have

$$|V_\alpha(x, t)| \leqslant k\bar{\rho} \quad \text{for} \quad 0 \leqslant x \leqslant 1 , \alpha k < t \leqslant T .$$

Therefore,

$$|v_1(x,t)| \leqslant \sum_{\alpha=0}^{t/k} |V_\alpha(x,t)| \leqslant k\bar{\rho} \cdot \frac{t}{k} = \bar{\rho}\, t ,$$

and, consequently,

$$|v(x,t)| \leqslant |v_0(x,t)| + |v_1(x,t)| \leqslant \bar{\rho}\, t + \max(\bar{f}, \bar{\varphi}, \bar{\psi}) ,$$

$$(0 \leqslant x \leqslant 1 , 0 \leqslant t \leqslant T) .$$

This completes the proof.

It follows immediately from the inequality (8.6) that the difference procedure in (2.2) is stable when the mesh ratio λ satisfies the criteria

$$(8.12) \qquad 0 < \lambda \leqslant \frac{1}{2} .$$

To see this, let $s = v^* - v$ denote the departure of the computed values from the exact solution of the difference problem as a result of errors not exceeding δ in numerical value at each mesh point. Then s is itself a solution of the problem formulated in (8.5) in which the upper bounds $\bar{f}$, $\bar{\varphi}$, $\bar{\psi}$, $\bar{\rho}$ satisfy the inequalities

$$\bar{f} \leqslant \delta , \quad \bar{\varphi} \leqslant \delta , \quad \bar{\psi} \leqslant \delta , \quad \bar{\rho} \leqslant \delta k^{-1} .$$

Therefore, the inequality (8.6) yields

$$|s(x, t)| \leqslant \delta(1 + \frac{t}{k}) ,$$

whence

$$(8.13) \qquad s = O(\frac{\delta}{k}) = O(\frac{\delta}{h^2}) .$$

The foregoing result no longer applies when the ratio $\lambda > 1/2$. There are examples that exhibit the instability of the finite difference equation (2.3) when $\lambda > 1/2$.

§9 An appraisal of the truncation error.

Let it be assumed that the solution $u(x, t)$ of the initial value problem (2.1) and its derivatives u_t and u_{xx} are continuous for $0 \leqslant x \leqslant 1$ and $t \geqslant 0$, and that the derivatives u_{tt} and u_{xxx} exist and are bounded in this region. Then, making use of Taylor's formula, and the fact that $u_t - u_{xx} = 0$, one obtains from (8.3) the expression

$$L_h\left[u\right] = \rho_1 k + \rho_2 h , \quad (0 < x < 1 , t > 0) , \tag{9.1}$$

where M_1 and M_2 are functions of the mesh such that

$$\begin{cases} |\rho_1| \leqslant M_1 \equiv \frac{1}{2}\,\ell.\text{u.b.}\,|u_{tt}(x, t)| , & (0 < x < 1, t > 0) \\ |\rho_2| \leqslant M_2 \equiv \frac{1}{3}\,\ell.\text{u.b.}\,|u_{xxx}(x,t)| , & (0 < x < 1, t > 0) . \end{cases} \tag{9.2}$$

Since the functions $u(x,t)$ and $v(x,t)$ satisfy the same initial and boundary values at the corresponding grid points, the truncation error $w = u-v$ is evidently a solution of the difference problem (8.5) in which the upper bounds $\bar{f}$, $\bar{\varphi}$, $\bar{\psi}$, $\bar{\rho}$ satisfy the relations

$$\bar{f} = \bar{\varphi} = \bar{\psi} = 0 , \quad \bar{\rho} = M_1 k + M_2 h .$$

Hence, if $0 < \lambda \leqslant \frac{1}{2}$, the inequality (8.6) yields the following appraisal for the truncation error

$$(9.3) \qquad |w| \leqslant t(M_1 k + M_2 h) \quad .$$

Since $k = \lambda h^2$, the truncation error will, for fixed λ, satisfy the relation

$$w = O(h) \quad \text{as} \quad h \longrightarrow 0 \ ,$$

provided that the solution u of the differential problem exists and its derivatives satisfy the assumed boundedness.

The appraisal in (9.3) also suffers from the fact that it requires a knowledge of the bounds of certain derivatives of the solution u of the differential problem. The question of convergence, however, is answered by the foregoing result if these bounds are merely known to exist.

Attention should be directed to the work of Juncosa and Young* on a problem of the same type considered here. They have established orders for the convergence of the difference solution that require only assumptions on the initial and boundary data.

Attention should also be called to the work of F. John** in connection with the finite difference approximation of a very general class of parabolic differential equations. In this work, the general partial differential

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*Juncosa and Young [9], [10].

**See F. John [11].

equation

$$u_t = a_0(x,t)u_{xx} + a_1(x,t)u_x + a_2(x,t)u + a_3(x,t)$$

is approximated by a difference equation of the type

$$v(x,t+k) = \sum_{r=-N}^{+N} c_r(x,t,h)\, v(x+rh,\ t) \quad .$$

Sufficient conditions are obtained under which a result analogous to that in (8.6) hold for these difference equations. A very simple criteria that admits of easy proof is that the coefficients $c_r(x, t, h)$ be non-negative for sufficiently small h .

§10 Numerical integration of a quasi-linear parabolic equation*

We consider the following non-linear, boundary value problem

$$(10.1)\quad \begin{cases} \dfrac{\partial^2 u}{\partial x^2} = F(x,t,u)\,\dfrac{\partial u}{\partial t} + G(x,t,u) \ , & (0 < x < 1,\ t > 0)\,, \\ u(0,\ t) = \varphi(t)\ ,\ u(1,\ t) = \psi(t)\ ,\ (t > 0)\,, & \\ u(x,\ 0) = f(x) \ , & (0 \leqslant x \leqslant 1)\ , \end{cases}$$

where $F \geqslant \rho > 0$. It is assumed that a solution of this problem exists in the closed region R: $0 \leqslant x \leqslant 1$, $0 \leqslant t \leqslant T$ such that

*The treatment in this section follows that given by Douglas in [12].

$\frac{\partial^4 u}{\partial x^4}$ and $\frac{\partial^2 u}{\partial t^2}$ exist and are bounded in R. Moreover, it is assumed that the functions $F(x,t,u)$ and $G(x,t,u)$ have bounded first partial derivatives with respect to u in R.

We introduce a mesh in the region R by setting

$$x_m = m\Delta x \ , \ (m = 0, 1, \ldots, M) \ , \qquad t_n = n\Delta t \ , \ (n = 0, 1, \ldots, T/\Delta t)$$

with $\Delta x = 1/M$. We then seek an approximation of the solution of (1) by replacing that system with the finite difference problem

$$(10.2)\begin{cases} \Delta_x^2 \, v_{m,n+1} = F(x_m, t_{n+1}, v_{n,m}) \Delta_t \, v_{m,n} + G(x_m, t_{n+1}, v_{mn}) \ ; \\ \qquad\qquad\qquad\qquad\qquad (m = 1, 2, \ldots, M-1; \ n \geqslant 0) \ , \\ v_{0,n} = \varphi(t_n) \ , \ v_{M,n} = \psi(t_n) \ , \qquad (n \geqslant 0) \ , \\ v_{m,0} = f(x_m) \ , \qquad (m = 1, 2, \ldots, M-1) \ , \end{cases}$$

where we have used the notation

$$(10.3)\begin{cases} \Delta_x^2 \, v_{m,n} = \dfrac{v_{m+1,n} - 2v_{m,n} + v_{m-1,n}}{(\Delta x)^2} \ , \\ \Delta_t \, v_{m,n} = \dfrac{v_{m,n+1} - v_{mn}}{\Delta t} \ . \end{cases}$$

The truncation error introduced at any mesh point as a result of replacing the system (10.1) by the difference equations (10.2) is the difference $w_{mn} = u_{mn} - v_{mn}$. In order to estimate the magnitude of this error in terms of the increments Δt and Δx, we proceed, as in the previous examples, to develop the difference equation of which w_{mn} is a solution. It is easily verified that

$$
(10.4) \quad \left\{
\begin{aligned}
\Delta_x^2 u_{m,n+1} &= u_{xx}(x_m, t_{n+1}) + \frac{(\Delta x)^2}{12} \bar{u}_{xxxx} \\
\Delta_t u_{m,n} &= u_t(x_m, t_{n+1}) - \frac{\Delta t}{2} \bar{u}_{tt} \\
F(x_m, t_{n+1}, u_{m,n}) &= F(x_m, t_{n+1}, u_{m,n+1}) - \bar{F}_u u_t(x_m, t_n) \Delta t - \\
&\qquad \frac{(\Delta t)^2}{2} \bar{F}_u \bar{u}_{tt} \\
G(x_m, t_{n+1}, u_{m,n}) &= G(x_m, t_{n+1}, u_{m,n+1}) - \bar{G}_u u_t(x_m, t_n) \Delta t - \\
&\qquad \frac{(\Delta t)^2}{2} \bar{G}_u \bar{u}_{tt} ,
\end{aligned}
\right.
$$

where the barred derivatives are evaluated at the intermediate points required by the mean value theorem. Substituting these in (10.1) yields the difference equation

$$
(10.5) \quad \Delta_x^2 u_{m,n+1} = F(x_m, t_{n+1}, u_{nm}) \Delta_t u_{mn} + G(x_m, t_{n+1}, u_{nm}) + g_{mn} ,
$$

where g_{mn} contains the terms in (10.4) with the factors Δt and Δx. As a consequence of the assumptions which have been made regarding the boundedness of the derivatives of the functions u, F, and G, the function g_{mn} is such that, in R,

$$(10.6) \qquad |g_{mn}| \leqslant K_1(\Delta t) + K_2(\Delta x)^2$$

for some positive constants K_1 and K_2.

If the first of equations (10.2) is subtracted from (10.5), we obtain

$$(10.7) \qquad \Delta_x^2 w_{m,n+1} = F(x_m, t_{n+1}, u_{mn}) \Delta_t u_{mn} - F(x_m, t_{n+1}, v_{mn}) \Delta_t v_{mn}$$

$$+ G(x_m, t_{n+1}, u_{mn}) - G(x_m, t_{n+1}, v_{mn}) + g_{mn} .$$

But, since $u_{mn} = v_{mn} + w_{mn}$, we can write

$$F(x_m, t_{n+1}, u_{mn}) = F(x_m, t_{n+1}, v_{nm}) + \bar{F}_u w_{mn} ,$$

$$G(x_m, t_{n+1}, u_{mn}) = G(x_m, t_{n+1}, v_{nm}) + \bar{G}_u w_{mn} .$$

Therefore (10.7) is equivalent to the equation

$$\Delta_x^2 w_{m\ n+1} = F(x_m, t_{n+1}, v_{nm}) \Delta_t w_{mn} + (\bar{F}_u \Delta_t u_{mn} + \bar{G}_u) w_{mn} + g_{mn} .$$

If we replace $\Delta_t w_{mn}$ by the difference quotient in (10.3), the equation can also be written as

$$\Delta_x^2 w_{m,n+1} - \frac{1}{\Delta t} F(x_m, t_{n+1}, v_{mn}) w_{m,n+1} \tag{10.8}$$

$$= - \frac{1}{\Delta t} F(x_m, t_{n+1}, v_{mn})(1-h_{mn}\Delta t) w_{mn} + g_{mn} ,$$

where

$$h_{mn} = \frac{\bar{G}_u - \bar{F}_u u_t(x_m, t_n) - \frac{\Delta t}{2} \bar{F}_u \bar{u}_{tt}}{F(x_m, t_{n+1}, v_{mn})} .$$

This is the desired difference equation for the truncation error w. Because of the supposed boundedness of the derivatives of u, F, and G, and the condition that $F \geqslant \rho > 0$ in R, it follows that the quantity h_{mn} is also bounded in R; that is, in R,

$$|h_{mn}| \leqslant A . \tag{10.9}$$

An estimate of the bounds for the magnitude of error $|w_{mn}|$ is readily obtained from the difference equation (10.8) by an application of the following two lemmas:

Lemma 10.1. If y_m is a solution of the difference problem

$$(10.10) \qquad \begin{cases} \Delta_x^2 y_m - \rho_m y_m = g_m , & (m = 1, 2, \ldots, M-1) \\ y_0 = y_M = 0 , & \end{cases}$$

and if $\rho_m > 0$ for all m, then

$$(10.11) \qquad \max_m |y_m| \leqslant \max_m \left| \frac{g_m}{\rho_m} \right| .$$

<u>Lemma</u> 10.2. If ϵ_n satisfies the recurrence relations

$$\epsilon_{n+1} \leqslant \beta \epsilon_n + \alpha , \qquad (n \geqslant 0) ,$$

where $\alpha > 0$ and $\beta \geqslant 1$, then

$$(10.12) \qquad \epsilon_n \leqslant \beta^n \epsilon_0 + n \alpha \beta^n , \qquad (n \geqslant 0) .$$

The proof of lemma 10.2 is by direct induction. The proof of lemma 10.1 is deferred until later.

Since $v_{mn} = u_{mn}$ initially and on the boundary, the truncation error $w_{mn} = u_{mn} - v_{mn}$ vanishes for $n = 0$, $m = 0$, and $m = M$. Therefore the result of lemma 10.1 holds for the solution of the difference equation (10.8). Note that $\rho_m = F(x_m, t_{n+1}, v_{nm}) \geqslant \rho$. The inequality corresponding to (10.11) can be written in the following form with the aid of (10.6) and (10.9)

$$\max_m |w_{m,n+1}| \leqslant (1+A\Delta t) \cdot \max_m |w_{mn}| + \frac{\Delta t}{\rho}\left[K_1 \Delta t + K_2(\Delta x)^2\right] \quad . \tag{10.13}$$

As in the previous sections, let

$$\lambda = \Delta t/(\Delta x)^2 = \text{const.} \tag{10.14}$$

Also, let ϵ_n denote the maximum numerical value of the truncation errors at the mesh points of the line $t = t_n$, i.e.,

$$\epsilon_n = \max_m |w_{mn}| \quad .$$

Then $\epsilon_0 = 0$ and, in accordance with (10.13) and (10.14), for all $\lambda \geqslant \lambda_0 > 0$ there exists a number $C > 0$ independent of λ such that

$$\epsilon_{n+1} \leqslant (1 + C\Delta t)\,\epsilon_n + C(\Delta t)^2 \quad .$$

Hence, by lemma 10.2,

$$\epsilon_n \leqslant C_n(1 + C\Delta t)^n (\Delta t)^2 = C\, t_n(1 + C\Delta t)^n \Delta t \ .$$

Since

$$(1 + C\Delta t)^n \leqslant e^{Cn\Delta t} = e^{Ct_n} ,$$

our result can be formulated as follows: For any mesh point (x, t) in the region R , $(0 \leqslant x \leqslant 1 , 0 \leqslant t \leqslant T)$, in which the functions u , F , and G and their derivatives satisfy the conditions of boundedness stipulated above, the magnitude of the truncation error is such that, for all $\lambda \geqslant \lambda_0 > 0$,

$$|w(x, t)| \leqslant C\, t\, e^{Ct} (\Delta t) \tag{10.15}$$

where $C > 0$ depends on the derivatives of u , F , and G and is independent of λ .

<u>Addendum to</u> §10. <u>Proof of Lemma</u> 10.2

The proof of lemma 10.2 consists of several parts. Let $g_m = g_m^+ + g_m^-$, where $g_m^+ \geqslant 0$, $g_m^- \leqslant 0$, and $g_m^+ \cdot g_m^- = 0$. Then $y_m = y_m^+ + y_m^-$, where $y_0^\pm = y_M^\pm = 0$ and $\Delta_x^2\, y_m^\pm - \rho_m\, y_m^\pm = g_m^\pm$, $(m = 1, 2, \ldots, M-1)$.

Consider first the function y_m^+ . It can be shown that $y_m^+ \leqslant 0$ for all m . For, suppose that $y_m^+ > 0$ for some m . Since $y_0^+ = y_M^0 = 0$, there is at least one value in the set $m = 1, 2, \ldots, M-1$ at which y_m^+ is a maximum. For this value of m , $\Delta^2\, y_m^+ \leqslant 0$ and therefore, since $\rho_m > 0$,

$$y_m^+ = \frac{1}{\rho_m} (\Delta_x^2 y_m^+ - g_m^+) \leqslant 0 .$$

This is a contradiction. At a negative minimum, $\Delta_x^2 y_m^+ \geqslant 0$ and

$$|y_m^+| = - y_m^+ = \frac{1}{\rho_m} (g_m^+ - \Delta_x^2 y_m^+) \leqslant \frac{g_m^+}{\rho_m} = \left| \left(\frac{g_m}{\rho_m}\right)^+ \right| .$$

Hence,

$$\max_m |y_m^+| \leqslant \max_m \left| \left(\frac{g_m}{\rho_m}\right)^+ \right| .$$

A similar argument serves to prove that maximum of $|y_m^-|$ satisfies the same inequality. Therefore, combining these results, we have

$$\max_m |y_m| = \max \left(\max_m |y_m^+| , \max_m |y_m^-|\right) \leqslant \max \left(\max_m \left| \left(\frac{g_m}{\rho_m}\right)^+ \right| , \max_m \left| \left(\frac{g_m}{\rho_m}\right)^- \right| \right)$$
$$= \max_m \left| \frac{g_m}{\rho_m} \right| .$$

This completes the proof of the lemma.

The following extension of lemma 10.1 is useful in establishing the stability of the difference procedure.

<u>Lemma</u> 10.3. If y_m is a solution of the difference problem

$$(10.6) \qquad \begin{cases} \Delta_x^2 y_m - \rho_m y_m = g_m , \\ y_0 = \varphi , \qquad y_M = \psi , \end{cases}$$

and if $\rho_m > 0$ for all m, then

$$\max |y_m| \leqslant \max_m \left| \frac{g_m}{\rho_m} \right| + 2 \max (|\varphi|, |\psi|) .$$

Proof. Define

$$z_m = y_m - \varphi - (\psi - \varphi) x_m , \qquad (m = 0, 1, 2, \ldots, M) .$$

If y_m is a solution of (10.6), z_m is a solution of the difference problem

$$\Delta_x^2 z_m - \rho_m z_m = g_m + \rho_m \left[\varphi + (\psi - \varphi) x_m \right] ,$$

$$z_0 = z_M = 0 .$$

Consequently, by (10.11),

$$\max_m |z_m| \leqslant \max_m \left| \frac{g_m}{\rho_m} \right| + \max_m \left| \varphi + (\psi - \varphi) x_m \right|$$

$$= \max_m \left| \frac{g_m}{\rho_m} \right| + \max_m (|\varphi|, |\psi|) .$$

Hence,

$$\max_m |y_m| \leqslant \max_m |z_m| + \max_m \left| \varphi + (\psi - \varphi) x_m \right|$$

$$\leqslant \max_m \left| \frac{g_m}{\rho_m} \right| + 2 \max (|\varphi|, |\psi|) .$$

§11 Stability of the non-linear finite difference equations (10.2)

Let v^* denote an approximation of the solution of the system (10.2) obtained by machine calculation. In particular, let v^* be a solution of the difference equations

$$
(11.1)\quad \begin{cases}
\Delta_x^2 v^*_{m,n+1} = F(x_m, t_{n+1}, v^*_{n,m}) \Delta_t v^*_{m,n} + G(x_m, t_{n+1}, v^*_{m,n}) + \delta_{m,n} , \\
\qquad\qquad (m=1, 2, \ldots, M-1;\ n \geqslant 0) , \\
v^*_{0,n} = \varphi(t_n) + \delta_{0,n},\ v^*_{M,n} = \psi(t_n) + \delta_{M,n} , \qquad (n \geqslant 0) , \\
v^*_{m,0} = f(x_m) + \delta_{m,0} , \qquad (m = 1, 2, \ldots, M-1) ,
\end{cases}
$$

where $\delta_{m,n}$ represents the residual error at the mesh point x_{mn}. It is not difficult to obtain an estimate of the magnitude of the departure $s = v^* - v$ of the computed solution from the exact solution when the conditions for the convergence of the system (10.2) are satisfied. There remains only to show that the difference equation which the departure s satisfies is not very different from that in (10.8) from which the inequality in (10.15) was obtained.

If equations (10.2) are subtracted from the corresponding equations in (11.1), one obtains, in the notation of equation (10.8),

$$
(11.2)\quad \begin{cases}
\Delta_x^2 s_{m,n+1} = F(x_m, t_{n+1}, v^*_{m,n})\Delta_t s_{m,n} + F(x_m, t_{n+1}, v_{m,n})\Delta_t v_{m,n} \\
\qquad + \bar{G}_u s_{m,n} + \delta_{m,n} , \qquad (m=1, 2, \ldots, M-1;\ n \geqslant 0) , \\
s_{0,n} = \delta_{0,n} , \quad s_{M,n} = \delta_{M,n} , \qquad (n \geqslant 0) , \\
s_{m,0} = \delta_{m,0} \qquad (m = 1, 2, \ldots, M-1) .
\end{cases}
$$

Now

$$\Delta_t v_{m,n} = \Delta_t w_{m,n} - \Delta_t u_{m,n} = \Delta_t w_{m,n} - u_t(x_m,t_n) - \frac{\Delta t}{2} \bar{u}_{tt} .$$

But, for $\Delta t/(\Delta x)^2 = \lambda$ with λ fixed, the inequality (10.15) implies that $w_{m,n} = O(\Delta t)$, $\Delta_t w_{mn} = O(1)$, and, consequently, $\Delta_t v_{m,n} = O(1)$ as $\Delta t \longrightarrow 0$. Since the residuals δ_{mn} are $O(1/\Delta t)$ as $\Delta t \longrightarrow 0$, it is evident that the system (11.2) can be cast in the form

$$(11.3) \quad \begin{cases} \Delta_x^2 s_{m,n+1} - \frac{1}{\Delta t} F(x_m,t_{n+1},v^*_{m,n}) s_{m,n+1} \\ \qquad = - \frac{1}{\Delta t} F(x_m,t_{n+1},v^*_{m,n}) \left[1 - h_{m,n} \Delta t\right] + g_{mn} , \\ s_{0,n} = g_{0,n} , \quad s_{m,0} = g_{M,n} , \\ s_{m,0} = g_{m,0} , \end{cases}$$

where for some positive constants A and B

$$|h_{m,n}| < A \quad \text{and} \quad |g_{mn}| < B(\Delta t)^{-1} .$$

If, as before, ϵ_n denotes the maximum numerical value of the error at the mesh points of the line $t = t_n$, but in this case as a result of round-off, then, according to 11.13 and lemmas (10.1) and (10.3), it readily follows that

$$\epsilon_n = 0(1/\Delta t) \quad \text{as} \quad \Delta t \longrightarrow 0 \ , \tag{11.4}$$

for $0 \leqslant t \leqslant T$. The procedure is therefore stable.

IV. HYPERBOLIC EQUATIONS

§12 Finite difference method for a hyperbolic system of partial differential equations*

Let us consider the initial value problem for the system of linear, first order, partial differential equations on the infinite interval:

$$(12.1)\quad \begin{cases} \vec{u}_t\,(x,\ t) = A\,\vec{u}_x\,(x,\ t) + B\,\vec{u}\ , & (-\infty < x < +\infty;\ t > 0)\ , \\ \vec{u}\,(x,\ 0) = \vec{f}\,(x)\ , & (-\infty < x < +\infty)\ , \end{cases}$$

where $\vec{u}$ is a column vector and A and B are $n \times n$ matrices. It is well known that an initial value problem for a higher order linear differential equation (or a system of higher order differential equations) can be reduced to an initial value problem for a first order system**. For example, the initial value problem for the second order differential equation:

$$u_{tt} = u_{xx}\ , \qquad (-\infty < x < \infty;\ t > 0),$$

$$u(0,\ x) = f(x)\ ,\ u_t(0,\ x) = g(x)\ , \qquad (-\infty < x < \infty)\ ,$$

---

*See Lax and Keller [13]. See also the paper by Lax and Richtmyer [14] in which both explicit and implicit difference systems are treated.

**See, for example, Petrovsky [1] p. 16.

is equivalent to the following:

$$u_t^{(1)} = u^{(2)} , \quad u_t^{(2)} = u_x^{(3)} , \quad u_t^{(3)} = u_x^{(2)} , \quad (-\infty < x < \infty; \; t > 0) ,$$

$$u^{(1)}(x,0) = f(x) , \quad u^{(2)}(x,0) = g(x) , \quad u^{(3)}(x,0) = f'(x) , \quad (-\infty < x < \infty) ,$$

where

$$u^{(1)} = u , \quad u^{(2)} = u_t , \quad u^{(3)} = u_x .$$

It will be supposed that the system of partial differential equations in (12.1) is of hyperbolic type. By this is meant that there exists a real, non-singular matrix P such that

$$\text{(12.2)} \qquad P A P^{-1} = D = \operatorname{dia}(d_1, d_2, \ldots, d_n) ,$$

where D is a diagonal matrix; the diagonal elements d_i of the matrix D are the so-called characteristic directions of the system. For the purposes of illustration, it will be supposed that the coefficient matrix A is constant. The elements of the matrix P are then also constants. The results in this simple case are typical of those in the more general case in which the elements of A are functions of x and t. It will also be assumed that a solution $\vec{u}$ of the initial value problem (12.1) exists such that $\vec{u}_{xx}$ and $\vec{u}_{tt}$ exist and are bounded for $t \geqslant 0$.

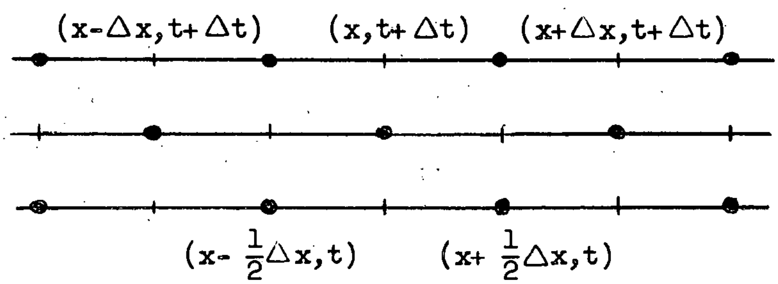
As an approximation of the system of partial differential equations in (12.1), we choose the system of difference equations:

$$\vec{v}(x,t+\Delta t) = \frac{\vec{v}(x+\frac{1}{2}\Delta x,t) + \vec{v}(x-\frac{1}{2}\Delta x,t)}{2} + \Delta t \left[A \frac{\vec{v}(x+\frac{1}{2}\Delta x,t) - \vec{v}(x-\frac{1}{2}\Delta x,t)}{\Delta x} \right.$$

(12.3)

$$\left. + B \frac{\vec{v}(x+\frac{1}{2}\Delta x,t) + \vec{v}(x-\frac{1}{2}\Delta x,t)}{2} \right] ,$$

where the values of the vector $\vec{v}(x, t)$ are computed at the mesh points of a staggered grid.



At the mesh points of the initial line it is required that

(12.4) $$\vec{v}(x, 0) = \vec{f}(x) .$$

It is evident that this constitutes an explicit difference method. The components of the vector $\vec{v}$ at any mesh point $(x, t + \Delta t)$ are explicitly given in terms of the previously computed values at points on the line t.

The truncation error in this case is the vector difference $\vec{w} = \vec{u} - \vec{v}$. The procedure for estimating the magnitude of this error as a function of the mesh size is, in principle, the same as in the cases already considered. We first derive the difference problem of which $\vec{w}$ is a solution. To this end, note that, because of the assumed properties of the vector function $\vec{u}$, we have

$$\frac{\vec{u}(x+\frac{\Delta x}{2},t)-\vec{u}(x-\frac{\Delta x}{2},t)}{\Delta x} = \vec{u}_x(x,t) + \frac{1}{8}(\vec{u}_{xx}' - \vec{u}_{xx}'')\Delta x ,$$

$$\frac{\vec{u}(x+\frac{\Delta x}{2},t)+\vec{u}(x-\frac{\Delta x}{2},t)}{2} = \vec{u}(x,t) + \frac{1}{16}(\vec{u}_{xx}' + \vec{u}_{xx}'')(\Delta x)^2 ,$$

$$\frac{\vec{u}(x, t+\Delta t) - \vec{u}(x, t)}{\Delta t} = \vec{u}_t(x, t) + \frac{1}{2}\vec{u}_{tt}'\Delta t ,$$

where the primed derivatives are evaluated at intermediate points in accordance with Taylor's remainder formula. Substituting these expressions in the first of (12.1) yields

$$\vec{u}(x,t+\Delta t) = \frac{1}{2}\left[I + 2A\frac{\Delta t}{\Delta x} + B\Delta t\right]\vec{u}\left(x+\frac{\Delta x}{2},t\right) + \frac{1}{2}\left[I - 2A\frac{\Delta t}{\Delta x} + B\Delta t\right] \tag{12.5}$$

$$\times\,\vec{u}\left(x-\frac{\Delta x}{2},t\right) + \vec{\alpha}(\Delta x)^2 + \vec{\beta}(\Delta t)^2 + \vec{\gamma}\,\Delta x\,\Delta t\ ,$$

where I is the identity matrix, and $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\gamma}$ are vectors depending on the values of the partial derivatives $\vec{u}_{tt}$ and $\vec{u}_{xx}$. The components of these vectors are therefore bounded for $t \geqslant 0$. On subtracting equation (12.3) from (12.5), we obtain

$$\vec{w}(x,t+\Delta t) = \frac{1}{2}\left[I + 2A\frac{\Delta t}{\Delta x} + B\Delta t\right]\vec{w}\left(x+\frac{\Delta x}{2},t\right) + \frac{1}{2}\left[I - 2A\frac{\Delta t}{\Delta x} + \right. \tag{12.6}$$

$$\left. + B\Delta t\right]\vec{w}\left(x-\frac{\Delta x}{2},t\right) + \vec{\alpha}(\Delta x)^2 + \vec{\beta}(\Delta t)^2 + \vec{\gamma}\,\Delta x\,\Delta t\ .$$

Moreover, since $\vec{v} = \vec{u}$ on the initial line,

$$\vec{w}(x, 0) = 0\ . \tag{12.7}$$

These are the equations defining the generation of the truncation error.

For the purposes of convenience in effecting an estimate of the magnitude of the truncation error, we define corresponding norms of a vector and a matrix relative to the matrix P given in (12.2). To be precise, if $(\vec{z})_i$ denote the ith element of the vector $\vec{z}$, the norm of $\vec{w}$ will be defined as

(12.8) $$\|\vec{w}\| = \max_i |(P\vec{w})_i| \quad .$$

If $(C)_{ij}$ denotes the elements of the matrix C, then the norm of the matrix A is defined as

(12.9) $$\|A\| = \max_i \sum_j |(P A P^{-1})_{ij}| \quad .$$

The definitions of the vector and matrix norms are such that

$$\|A\vec{w}\| \leqslant \|A\| \cdot \|\vec{w}\| \quad .$$

Then a measure of the magnitude of the truncation error at the <u>nth</u> step of the process, that is, for $t = n\Delta t$, will be given by the quantity

(12.10) $$\epsilon_n = \max_x \|\vec{w}(x, t)\| \quad ,$$

where x ranges over all the mesh points on the line $t = n\Delta t$.

Let us choose the mesh ratio $\Delta t/\Delta x$ so that

(12.11) $$\Delta t/\Delta x \leqslant 1/2c \quad ,$$

where

(12.12) $$c = \max_i |d_i| \quad .$$

Then all elements of the matrices

$$S\left(I \pm 2A \frac{\Delta t}{\Delta x}\right) S^{-1} = \left(I \pm 2D \frac{\Delta t}{\Delta x}\right)$$

are non-negative. Since

$$\left\| \vec{w}\left(x \pm \frac{\Delta x}{2}, t\right)\right\| \leqslant \epsilon_n ,$$

it readily follows from these facts that

$$\left\| \frac{1}{2}\left(I + 2A \frac{\Delta t}{\Delta x}\right) \vec{w}\left(x + \frac{\Delta x}{2}, t\right) + \frac{1}{2}\left(I - 2A \frac{\Delta t}{\Delta x}\right) \vec{w}\left(x - \frac{\Delta x}{2}, t\right)\right\| \leqslant \epsilon_n .$$

This being true, it is immediately evident from equation (12.6) that there exists a number ρ such that

$$\text{(12.13)} \qquad \epsilon_{n+1} \leqslant (1+\rho\Delta t)\, \epsilon_n + \rho\left[(\Delta x)^2 + (\Delta t)^2 + \Delta x\, \Delta t\right] ;$$

the number ρ depends on the upper bounds of $\|B\|$ and the corresponding norms of the second order partial derivatives which appear in the vectors $\vec{\alpha}$, $\vec{\beta}$, and $\vec{\gamma}$. According to lemma (10.2), the recurrence relation in (12.13) implies that

$$\epsilon_n \leqslant (1+\rho\Delta t)^n \epsilon_0 + \rho n(1+\rho\Delta t)^n \left[(\Delta x)^2 + (\Delta t)^2 + \Delta x\, \Delta t\right] ,$$

and therefore

(12.14) $$\epsilon_n \leqslant \epsilon_0 e^{\rho t} + \rho\, t\, e^{\rho t} \left[\frac{(\Delta x)^2}{\Delta t} + \Delta t + \Delta x \right] .$$

Let ratio $\lambda = (\Delta t)/(\Delta x)$ be held fixed during the procedure. Then, in view of (12.7), we have shown that when

$$\lambda \leqslant \frac{1}{2c}$$

the magnitude of the truncation error is such that*

(12.15) $$\epsilon_n \leqslant \rho(\lambda^2 + \lambda + 1)\, t\, e^{\rho t} \Delta t \quad , \quad (n = t/\Delta t) ,$$

whence, for $0 \leqslant t \leqslant T$,

$$\epsilon_n = O(\Delta t) \quad \text{as} \quad \Delta t \longrightarrow 0 .$$

Therefore, under the supposed conditions on the solution of the differential system (12.1), the difference procedure (12.3) converges.

The inequality (12.4) also serves as a means of estimating the magnitude of the error in the approximate solution that stems from rounding-off of the computed quantities. In this instance, let ϵ_0 denote the maximum numerical value of the errors introduced by this source at the mesh points of any line $t = t_0$, and suppose that all subsequent

---

*Note that since the matrix P is nonsingular the norm $\| \vec{w} \|$ can vanish if and only if $|\vec{w}| = 0$. Hence, $\epsilon_n \longrightarrow 0$ if and only if $\vec{w} \longrightarrow 0$.

calculations are exact. The departure s in (4.1) then satisfies the difference equation (12.6) with $\vec{\alpha} = \vec{\beta} = \vec{\gamma} = 0$ for $t > t_0$. Consequently, the maximum numerical value of the departure at the nth subsequent step, which we again denote by ϵ_n, does not exceed the first term in (12.14):

$$\epsilon_n \leqslant \epsilon_0 e^{\rho t} , \qquad (t = t_0 + n \Delta t) ,$$

where ρ has the same meaning as before. The departure resulting from an accumulation of the round-off errors introduced at each of n steps will therefore not exceed, in numerical value, the number

$$\epsilon_0 n e^{\rho t} = \frac{\epsilon_0 t}{\Delta t} e^{\rho t} , \qquad (t = n \Delta t) .$$

Consequently, the departure s satisfies the following relation in any interval $0 \leqslant t \leqslant T$:

$$|s| \leqslant O\left((\Delta t)^{-1}\right) . \tag{12.16}$$

Hence, by the criteria adopted earlier, the difference procedure (12.3) is stable.

BIBLIOGRAPHY

1. I. G. Petrovsky, Lectures on partial differential equations, Interscience Publishers, 1954.

2. R. Courant and I. Hilbert, Methoden der mathematishen Physik. Vol. I, II, Interscience Publishers, 1943.

3. J. Hadamard, Lectures on Cauchy's problem, New Haven, 1923.

4. T. S. Motzkin and W. Wasow, On the approximation of linear elliptic differential equations by difference equations with positive coefficients, Jour. Math. Phys. 31, 253-259 (1953).

5. W. Wasow, Discrete approximations to elliptic differential equations, Jour. Appl. Math. and Phys. (ZAMP), 6, 81-97 (1955).

6. S. Gerschgorn, Fehlerabschätzung für Differenzenverfahren zur Lösung partieller Differentialgleichungen, Z. Angew. Math. Mech., 10(4), 373-382 (1930).

7. L. Collatz, Bemerkungen zur Fahlerabschätzung für das Differenzenverfahren bei partiellen Differentialgleichungen, Z. Angew. Math. Mech., 13, 56-57 (1933).

8. P. C. Rosenbloom, The difference equation method for solving the Dirichlet problem, Nat. Bur. Standards Appl. Math. Series, 18, 231-237 (1952).

9. M. L. Juncosa and D. M. Young, On the convergence of a solution of a difference equation to a solution of the equation of diffusion, Proc. Amer. Math. Soc., 5, 168-174 (1954).

10. M. L. Juncosa and D. M. Young, On the order of convergence of solutions of a difference equation to a solution of the diffusion equation, Jour. S. I. A. M., 1, 111-135 (1953).

11. F. John, On the integration of parabolic equations by difference methods, Comm. Pure and Appl. Math., 5, 155-211 (1952).

12. J. Douglas, Jr., On the numerical integration of quasi-linear parabolic difference equations, to appear in Pacific Jour. Math.

13. P. D. Lax and J. B. Keller, The initial and mixed initial and boundary value problem for hyperbolic systems, Los Alamos Report LAMS-1205.

14. P. D. Lax and R. D. Richtmyer, Survey of the stability of linear finite difference equations, Comm. Pure and Appl. Math., 9, 267-293 (1956).

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