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ARCH AND DEVELOPM

DYNAMICS OF THE SUPERCRITICAL WATER REACTOR

A Report to the
OAK RIDGE NATIONAL LABORATORY
from
NUCLEAR DEVELOPMENT ASSOCIATES, INC.

OAK RIDGE NATIONAL LABORATORY
OPERATED BY
CARBIDE AND CARBON CHEMICALS COMPANY
A DIVISION OF UNION CARBIDE AND CARBON CORPORATION

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DYNAMICS OF THE SUPERCRITICAL WATER REACTOR

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SUMMARY

The background work on the dynamics and control of the Supercritical Water Reactor described in ORNL-1177 is summarized here with emphasis on the determination of the self or inherent stability of the machine. Variations in water density are found to provide a substantial amount of self regulation, and the reactor appears quite amenable to control under steady conditions.

The use of a fluid which undergoes a six-fold expansion in passing through the reactor for both moderating and cooling makes necessary a more elaborate treatment of the hydrodynamic behavior of this fluid than is usual. The equations of motion of the system are derived and simplified by linearization, thus limiting the validity of the results to the case of small departures from equilibrium. The resulting partial differential equations are then transformed by a variational procedure into a system of first order, ordinary differential equations. The time responses to some disturbances of interest are then determined. Eight major reactor periods, all stable, have been found. Their values are: 5.4 min., 21.9 sec., 1.60 sec., 0.646 sec., 0.110 sec., 0.0189 sec., 0.0087 sec., and 0.0045 sec.

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The variational method devised for this particular problem should be of general interest in the field of reactor control.

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Introduction

The work described in this report was carried out as part of the feasibility study (ORNL-1177) of a supercritical water reactor (SCWR) for use in nuclear propulsion of aircraft. The object of this work was to study the dynamic behavior of a particular design of supercritical water reactor. Numerical results are presented in Appendix I.

The basic reactor configuration considered herein is described in the next section. Also in that section, the idealizations of the actual physical system which were made for computational reasons, are discussed. In Section 2 the equations describing the idealized system are derived. The intricate and strong interaction of the flow and fission aspects of the SCWR will be seen by a consideration of these equations.

It is because of the complex and strong interaction of fission and water systems that the calculations reported herein are more tedious and complicated than normal in the reactor field. These computations are described in Sections 3 through 5 below. The mathematical processes used in these sections are of general applicability, even

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though they are described in respect to the specific problem of the SCWR.

Section 6 is concerned with consideration of a simpler reactor model so as to obtain insight into the behavior of the various components of the SCWR and so as to compute the effects of some design changes on the dynamics of SCWR.

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Section 1. DESCRIPTION OF BASIC SYSTEM

The SCWR under consideration herein is a reactor with uranium-bearing stainless steel fuel elements (ORNL-1177, page 39) cooled, moderated, and reflected with water at super-critical pressures. The water enters the reactor at a temperature below critical and leaves with a temperature above critical. The relevant operating characteristics of the reactor are given in Appendixes II and III.

Water entering the reactor is split into two streams - one to the moderator region and the other directly to the fuel elements, which are immersed in the moderating water. The relative proportion of water entering these streams is adjustable so as to obtain shim control. The water which passes through the moderating region is mixed with the stream flowing directly to the fuel elements at the entrance to the fuel elements. Thus, all water leaving the reactor first flows through the fuel elements.

The fuel elements are all the same and are uniform along their length. These elements are distributed with cylindrical symmetry in such a manner as to cause each element

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to generate nearly the same power.

The water entering the reactor comes via a pump from the condensers, passes through the thermal shield before entering the reactor, and finally exits into a steam turbine.

The system described above is idealized in several ways -- for the neutronics, for the flow, and for the external system. The inlet to the reactor is taken to be a constant pressure, constant temperature source of water. The outlet is considered to be a constant pressure receiver.

In studying the water flow through the reactor, the case in which all the water flows first through the moderating region and then into the fuel tubes is considered. The pressure in the moderating region is assumed constant (equal to the inlet pressure). The validity of this approximation is discussed in ORNL-1177, page 101). In addition, the water in the moderating region is assumed well stirred, so that it has uniform properties and so that the water leaving the moderating region has the same properties as the water therein.

The fuel tubes are all assumed to have identical longitudinal power distributions, so that one may consider the flow and thermodynamic characteristics of the cooling streams to be the same for all streams. In effect, the group of all

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cooling streams is considered as a single entity. This idealized flow system is pictured schematically in Figure 1.

In studying the neutronics of the reactor, the two-group diffusion theory approximation is used. This is a relatively accurate approximation for dynamic considerations in a water moderated reactor. Two groups of delayed neutrons are considered. The fuel elements are considered to be distributed rather than discrete. The distribution in the radial direction (perpendicular to the axis of rotational symmetry of the system) is so chosen that the power density per unit uranium mass is uniform in the radial direction.* This implies that the thermal neutron flux is constant in the radial direction. No end reflectors are considered, so that the flux varies sinusoidally in the longitudinal direction, vanishing at the extrapolated

* In actuality the core is heterogeneous due to the discreteness of the fuel elements, and the degree of heterogeneity varies with distance from the core axis. The main effect of this coarseness of structure on the dynamics of the reactor occurs when the fuel elements are further apart than about one diffusion length of thermal neutrons in the moderator. For then the moderator absorbs an important fraction of moderated neutrons over and above what it would absorb in a homogeneous core, before they diffuse into the fuel element. An increase in power causes a decrease in moderator water density and an increase in its temperature. Both of these effects tend to increase the thermal diffusion length and permit more neutron absorptions by fuel elements, thus increasing the reactivity. This unstabilizing effect of coarseness is not included in this report. Calculations indicate that the magnitude of this effect is about 1/5 as large as the concomitant stabilizing density change.

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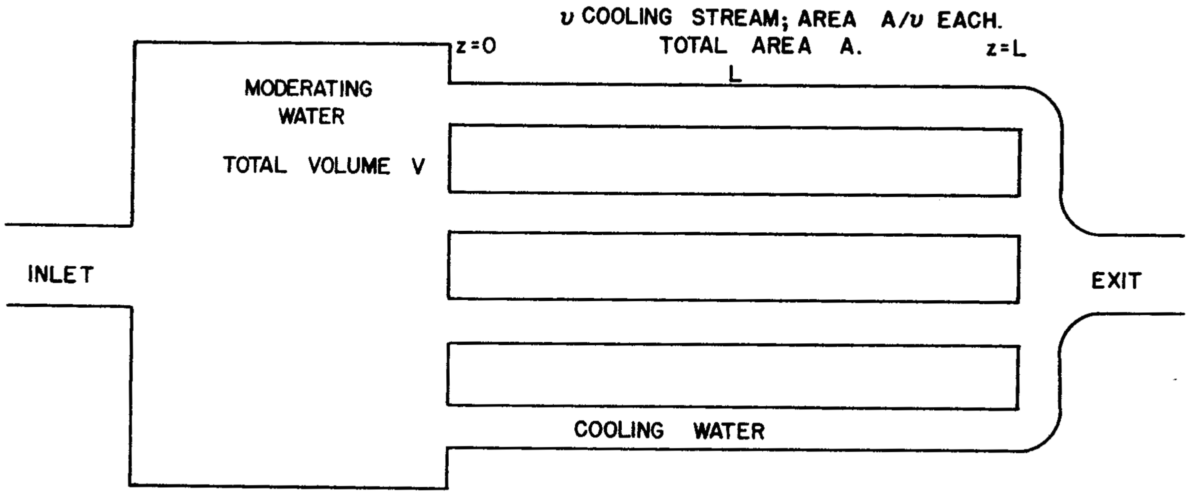
FIGURE I

SCHEMATIC REPRESENTATION OF BASIC IDEALIZED FLOW SYSTEM



PRESSURE	p_o	p_o	$p(z,t)$	p_1
TEMPERATURE	T_o	$T_m(t)$	$T(z,t)$	$T_1(t)$
DENSITY	ρ_o	$\rho_m(t)$	$\rho(z,t)$	$\rho_1(t)$
TOTAL MASS FLOW	$Ag_o(t)$		$Ag(z,t)$	$Ag_1(t)$

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end points of the fuel elements. The displacement of water by fuel elements is taken into account only by using the correct average density for the water. The outer boundary of the reflector is assumed to be a cylinder. The relevant steady state neutron data is given in Appendix II.

Further considerations are necessary relative to the transfer of the heat liberated in the fission process to the cooling and moderating water. Most of the heat of fission appears directly in the fuel elements as arising from kinetic energy of fission fragments and also of beta decay particles. In addition, some gamma heating appears directly in the fuel elements. In considering the dynamics, the iron is assumed to be cooled only by the cooling water.

The remainder of the gamma ray heating and all of the kinetic energy lost by neutrons slowing down is assumed to heat the moderating water.

In evaluating the effect of water density and temperature changes on the neutron behavior of the reactor, average water densities and temperatures are used as if uniform throughout the reactor.

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Section 2. DERIVATION OF EQUATIONS

The conservation equations for thermal neutrons and delayed neutron emitters are given below, with the approximation that time taken for fission or delayed neutrons to slow down may be neglected compared to the thermal lifetime. In addition, since this is a two-group theory, the difference in slowing down distributions of delayed and fission neutrons is neglected. Then with the assumed uniform density water and the variable density fuel the equations are given by:

$$\frac{\partial n}{\partial t} = K_f\left[(1-\beta)\eta y n + \sum_i \lambda_i c_i\right] - (y + w\mathcal{X}_s)n \qquad (2.1)$$

$$\frac{\partial c_i}{\partial t} = -\lambda_i c_i + \beta_i \eta y n; \qquad i = 1, 2 \qquad (2.2i)$$

where $\mathcal{X}_s = -\alpha_s^2 \nabla^2 + 1$

$$K_f f(\underline{r}) = \int k_f(\underline{r}, \underline{r}') f(\underline{r}') d\underline{r}'$$

$$(-\alpha_f^2 \nabla^2 + 1) k_f(\underline{r}, \underline{r}') = \delta(\underline{r} - \underline{r}')$$

These equations are subject to the boundary conditions that $n(\underline{r})$ vanishes at the extrapolated boundary and that $K_f(\underline{r}, \underline{r}')$

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vanishes when $\underline{r}$ (or $\underline{r}'$) refers to the extrapolated boundaries.

The symbols used are defined below.

$\underline{r}$ = position vector,

t = time,

n = no. of thermal neutrons per unit volume,

c_i = no. of i'th delayed neutron emitters per unit volume,

β_i = fraction of fission neutrons which come from the i'th delayed neutron emitter,

$\beta = \Sigma\beta_i$,

λ_i = reciprocal mean life of the i'th delayed neutron emitter,

η = no. of neutrons resulting from the absorptions of one neutron by fuel (steel-uranium mixture),

y = reciprocal mean life of a thermal neutron for absorption by fuel,

w = reciprocal mean life of a thermal neutron for absorption by water,

α_s = thermal diffusion length in the water,

α_f = slowing down length in the reactor.

Of these quantities, n, c_i, and y vary with position; the remainder are independent of $\underline{r}$.

In equation (1), the expression $(1-\beta)\eta yn + \sum_i \lambda_i c_i$ represents the instantaneous source density of fast neutrons,

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both prompt and delayed. Thus $K_f[(1-\beta)\eta yn + \sum_i \lambda_i c_i]$ represents the slow neutron source density. The term $(y + w\mathcal{K}_s)n$ represents the slow neutron sink density by both absorption in fuel and water and by leakage. Similarly in equation (2), $\lambda_i c_i$ is the loss density of delayed neutron emitters by decay and $\beta_i \eta yn$ is the source density via fission.

The conservation equation for Xe^{135} is

$$\frac{\partial X}{\partial t} = I + bn - (\lambda_x + n\overline{v\sigma_x})X \tag{2.3}$$

where

X = atomic density of Xe,

I = source density of Xe atoms via decay of I^{135},

bn = source density of Xe atoms directly from fission,

λ_x = reciprocal mean decay time of Xe,

$\overline{v\sigma}_x$ = product of neutron velocity and Xe absorption cross section averaged over the thermal neutron spectrum.

According to the basic idealised flow system pictured in Figure 1, water of constant temperature and density enters the moderating chamber, is heated, well stirred, and passes out into identical cooling streams each having a flow area A/ν. The con-

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servation of mass in the moderating chamber requires that

$$V \frac{d\rho_m}{dt} = A[g_o - g(0)] \tag{2.4}$$

where

V = volume of moderating chamber,

ρ_m = average density of moderating water,

$g(0)$ = mass flow density at the inlet of the fuel tube,

Ag_o = total mass flow into moderating chamber.

The energy conservation equation for the moderating region can be written as

$$V \frac{d}{dt}(\rho_m u_m) \approx V \frac{d}{dt}(\rho_m h_m) \tag{2.5}$$

$$= q_m + A[g_o h_o - g(0)h_m]$$

where

u_m = average specific (per unit mass) internal energy of the moderating water,

h_o = specific enthalpy of the water entering the moderating region,

h_m = average specific enthalpy of the moderating water,

$\rho_m u_m = \rho_m h_m - p_o$,

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p_o = pressure of the moderating water; assumed constant,

q_m = total power into the moderating water, coming from neutron heating, γ-ray heating, and conduction heating-discussed below.

By the assumption of good mixing in the moderating region, ρ_m, u_m, and h_m are the uniform values of density, energy, and enthalpy throughout the moderator, except for a small region at the inlet.

For the cooling streams, the equations for conservation of mass, momentum, and energy, are now written:

$$\frac{\partial \rho}{\partial t} + \frac{\partial g}{\partial z} = 0 \tag{2.6}$$

$$-\frac{\partial p}{\partial z} = \frac{B}{\rho} g^m + \rho\left(\frac{\partial}{\partial t} + \frac{g}{\rho}\frac{\partial}{\partial z}\right)\left(\frac{g}{\rho}\right) \tag{2.7}$$

$$\rho\left(\frac{\partial}{\partial t} + \frac{g}{\rho}\frac{\partial}{\partial z}\right)h = \left(\frac{\partial}{\partial t} + \frac{g}{\rho}\frac{\partial}{\partial z}\right)p + (\theta - T)\frac{H}{A} \tag{2.8}$$

where ρ, g, p, h are the density, mass flow density, pressure and specific enthalpy of the water in the cooling stream.

T = temperature of the water,

θ = temperature of the steel (cooling stream wall),

$\frac{Bg^m}{\rho}$ = the friction pressure drop per unit length,

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$$H = g^{\mu}F(\theta + T).$$

Here H is the total heat transfer coefficient per unit length along the coolant stream direction. In the derivation of Eq. 2.8 explicit use is made of the previously mentioned assumption that all heat entering the cooling streams comes from the steel.

In equations (2.7) and (2.8) the expression $\frac{\partial}{\partial t} + \frac{g}{\rho}\frac{\partial}{\partial z}$ is simply the total time derivative $\frac{d}{dt}$; i.e. the rate of change of a property of the fluid as it moves along. Equation (2.8) can be derived by starting with

$$\frac{du}{dt} = (\theta - T)\frac{H}{A\rho} - p\frac{d}{dt}\frac{1}{\rho} \qquad (2.9)$$

where the second term on the right is the flow power. Using now, $h = u + p/\rho$, and replacing $p\frac{d}{dt}\frac{1}{\rho}$ by $\frac{d}{dt}\frac{p}{\rho} - \frac{1}{\rho}\frac{dp}{dt}$, equation (2.8) results.

Connecting the moderator equations with the coolant equations are the conditions

$$\rho_m = \rho(0) \qquad (2.10)$$

$$p_o = p(0) + \frac{1}{2}g^2(0)/\rho(0) \qquad (2.11)$$

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Equation (2.11) gives the pressure change according to Bernoulli's equation. In equation (2.10), the adiabatic expansion of the water across the interface is neglected. Further, there exists at the exit end of the coolant stream the boundary condition of a constant pressure (p_1) region of large extent into which the coolant flows.

$$p(L) + \frac{g^2(L)}{2\rho(L)} = p_1 \qquad (2.12)$$

In the equation for the steel temperature θ the small drop in temperature within the fuel element from center to wall is neglected since the major heat block arises from the steel to water film drop. Thus $C \frac{\partial\theta}{\partial t} = q_I - (\theta - T)H$ (2.13)

where

q_I = the total power into all of the steel per unit length along the coolant flow direction.

C = the total heat capacity of the steel per unit length along the coolant stream.

The interaction between the neutronics and the flow is now discussed. First is considered the effect of the neutronics on the flow (computations of q_m and q_I). The effect of the flow on the neutronics is treated later.

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The moderating water is heated by neutrons, gamma rays, conduction of heat from the fuel tubes, and conduction of heat from the pressure shell and thermal shield. The heat transmitted to moderator via neutrons and much of the heat transmitted to the moderator via gamma rays appears in the moderator almost immediately (times less than or of the order of the neutron lifetime). This is also true of the capture gammas. The remainder of the gamma ray heating, (which comes from gammas emitted by the fission products), the conduction heating from fuel tubes, and the heat from pressure shell and thermal shield, appears only at times long compared with the neutron lifetime. In the computations described below, the stabilizing influence of the delayed-moderator heating is neglected. Quantitatively, this neglect influences the behav-ior of the reactor in the first second after a disturbance very slightly, whereas it underestimates the eventual stabil-izing influence of the moderator on the reactor.

The iron is heated by the kinetic energy of the fission fragments and that portion of the prompt gammas, delayed gammas, and capture gammas, which are absorbed in the iron. In studying the dynamics, the changes in delayed gamma heating are neglected.

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Because of the assumption of a well stirred moderator, all that is needed is the total heat into the moderator. For the heat into the iron, it is necessary to know not only the total amount but also the distribution. The distribution of direct fission heating is easily computed. A computation of the gamma heating distribution is somewhat more difficult. Since this gamma heating is a small fraction of the total heating, for convenience it was taken as uniformly distributed in the iron. Appendix IV contains all gamma heating and neutron results.

In accordance with the above remarks we have

$$q_m = q_m^{(p)} + q_m^{(d)} \tag{2.14}$$

$$q_m^{(p)} = q_o \int n d\underline{r} \tag{2.15}$$

where $q_m^{(p)}$ is the prompt heating of the moderator and is proportional to the total power as is evidenced by equation (2.15).

Further

$$q_I = q_1 n + q_2 \int n d\underline{r} \tag{2.16}$$

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where q_1n is the direct fission heat*, being proportional to the local power, and $q_2\int n d\underline{r}$ is the gamma heating (assumed prompt).

The effects of the flow system on the neutron system is implicitly contained in equations (2.1) and (2.2) where the parameters η, y, w, α_s and α_f are functions of the temperature and density of the water. In the evaluation of these parameters, a single average water density and a single average water temperature for moderating and cooling water is used. In addition, the thermal neutrons are characterized by a temperature equal to the average water temperature. In accordance with these remarks we write:

$$\eta = \eta_o \frac{\overline{v\Sigma_U}}{\overline{v\Sigma_U} + \overline{v\Sigma_I} + \overline{v\sigma_x}\, X} \qquad (2.17)$$

$$y = \overline{v\Sigma_U} + \overline{v\Sigma_I} + \overline{v\sigma_x}\, X \qquad (2.18)$$

where

η_o = number of fission neutrons resulting from the absorption of one neutron in uranium,

$\overline{v\Sigma_U}$ = product of neutron velocity and absorption cross-section of uranium per unit volume, averaged over the thermal neutron spectrum,

*In the product q_1n, the factor n is to be viewed as a function of z only. This is legitimate because n is radially flat in the core. [redacted]

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$\overline{v\Sigma_I}$ = same for iron,

$\overline{v\sigma_x}X$ = same for xenon.

$$w = (\bar{\rho}/\rho_o)\ \overline{v\Sigma_w^{o}} \tag{2.19}$$

$$\alpha_s^2 = (\rho_o/\bar{\rho})^2\ (\alpha_s^o)^2 \tag{2.20}$$

$$\alpha_f^2 = (\rho_o/\bar{\rho})^2\ (\alpha_f^o)^2 \tag{2.21}$$

where

$\bar{\rho}$ = average mass density of all water in the reactor (moderator and fuel-tube water),

$\overline{v\Sigma_w^{o}}$ = product of neutron velocity and water absorption cross-section per unit volume, averaged over the thermal neutron spectrum, when the water density is ρ_o, and is a function of the neutron temperature $\bar{T}$,

α_s^o = thermal diffusion length of neutrons in water of density ρ_o and is a function of the neutron temperature $\bar{T}$,

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α_f^o = fast diffusion length in water of density ρ_o, and is practically independent of neutron temperature $\overline{T}$.

The average water density $\overline{\rho}$ is given by

$$\overline{\rho} = \frac{\rho_m V + A\int_0^L \rho dz}{V + AL} \tag{2.22}$$

and the neutron temperature is taken as the average temperature of the water.

$$\overline{T} = \frac{\rho_m T_m V + A\int_0^L \rho T dz}{\rho_m V + A\int_0^L \rho dz} \tag{2.23}$$

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Section 3. THE LINEARIZED SYSTEM

The equations of motion of the system as presented in the preceding section are non-linear partial differential equations. The only general method of solution is one of numerical integration. Such a process is not justified for the problem under consideration. Rather, if considerations are limited to only small departures of the system from its steady state (equilibrium) configuration, a much easier mathematical problem results. This is the solution of linearized equations.

It should be pointed out that stability of the system under small oscillations as considered here is a necessary condition for the stability of the system. On the other hand, such stability under small oscillations is not sufficient to assure stability of the system, since the possibility of large unstable oscillations characteristic of non-linear systems is not ruled out. Such large oscillations which can exist within a system stable for small oscillations are more appropriately part of the study of possible accidents than part of the study of stability and reactor dynamics. At any rate, this type of difficulty is not considered further in

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this report.

Such linearized equations arise from the equations of Section 2 by the following procedure. Each variable x occuring in the equations, except the position and time, is expressed as the sum of its equilibrium or steady state value x^{o} and its departure from this equilibrium value, x'. Each term in the equation being linearized is expanded in a Taylor series in the various quantities x'. In such expansions, the constant and linear terms are considered, all other terms being neglected. The constant terms by themselves give rise to the conditions for an equilibrium configuration. The linear terms then describe the small oscillations about equilibrium. The higher order terms which are neglected are important for large oscillations.

A few illustrative examples of the linearization process are given in Table 3.1.

TABLE 3.1

Examples Of The Linearization Process

Unlinearized		Steady State	First Order	
$f(x,t)$	$=$	$f(x^{o})$	$+ \left(\frac{\partial f}{\partial x}\right)^{o} x'$	*
$f(x,y,t)$	$=$	$f(x^{o},y^{o})$	$+ \left(\frac{\partial f}{\partial x}\right)^{o} x' + \left(\frac{\partial f}{\partial y}\right)^{o} y'$	**
$f(x,y,t)\, \frac{\partial}{\partial t}\, g(x,y,t)$	$=$	0	$+ f(x^{o},y^{o}) \left[\left(\frac{\partial g}{\partial x}\right)^{o} \frac{\partial x'}{\partial t} + \left(\frac{\partial g}{\partial y}\right)^{o} \frac{\partial y'}{\partial t}\right]$	
$\frac{\partial}{\partial z} f(u,v,t)$	$=$	$\left[\left(\frac{\partial f}{\partial u}\right)^{o} \frac{\partial u^{o}}{\partial z} + \left(\frac{\partial f}{\partial v}\right)^{o} \frac{\partial v^{o}}{\partial z}\right]$	$+ \frac{\partial}{\partial z}\left[\left(\frac{\partial f}{\partial u}\right)^{o} u' + \left(\frac{\partial f}{\partial v}\right)^{o} v'\right]$	

$$* \quad \left(\frac{\partial f}{\partial x}\right)^{o} = \left.\frac{\partial f}{\partial x}\right|_{x=x^{o}}$$

$$** \quad \left(\frac{\partial f}{\partial x}\right)^{o} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x^{o}\\ y=y^{o}}}$$

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Before writing down the linearized equations as obtained from the non-linearized equations of Section22, it is useful to indicate explicitly the basic variables and how such quantities as η, y, w, etc. are related to them. In this manner, the various partial derivatives arising in the linearization process are made specific.

The fundamental variables are $\underline{r}$, the position in the reactor (or z, the position along a fuel tube) and t, the time. The basic thermodynamic variables are $\rho_m(t)$, $\rho(z,t)$, $p(z,t)$, and $\theta(z,t)$; while $n(\underline{r},t)$ is the basic neutron variable.. Other quantities will be considered as functions of these variables either directly or indirectly as tabulated below, where the quantity in question is an explicit function of the variables or quantities inside the parentheses.

$T(\rho)$		(this neglects small effect of pressure changes)
$T_m(\rho_m)$		
$\overline{\rho}(\rho_m,\rho)$		
$\overline{T}(\rho_m,\rho)$	;	$h(\rho,p)$
*$\eta y(\overline{T})$		$h_m(\rho_m)$
*$y(X,r)$		
*$w(\overline{\rho})$		
$\alpha_f(\overline{\rho})$		$K_f(\alpha_f)$
$\alpha_s(\overline{\rho},\overline{T})$		$\mathcal{L}_s(\alpha_s)$

* This assumes $\overline{\sigma v}$ independent of temperature. For our operating point, it is approximately true for xenon.

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The linearized equations are now written.

$$
\begin{aligned}
\partial n'/\partial t = {} & \left[K_f(1-\beta)\eta y - y - w\varkappa_s\right]^o n' + \Sigma_i\, \lambda_i K_f{}^o c_i{}' - (y_x n)^o X' \\
& + \Bigg\{ \left[K_{f\alpha_f}(1-\beta)\eta y n \alpha_{f\bar{\rho}} - w_{\bar{\rho}}\varkappa_s n - w\varkappa_{s\alpha_s}\alpha_{s\bar{\rho}} n\right] \frac{V_m}{V_m + AL} \\
& \quad - w\varkappa_{s\alpha_s}\alpha_{s\bar{T}}\, \bar{T}_{\rho m} n \Bigg\}^o \rho_m{}' \\
& + \Bigg\{ \left[K_{f\alpha_f}(1-\beta)\eta y n \alpha_{f\bar{\rho}} - w_{\bar{\rho}}\varkappa_s n \right. \\
& \quad \left. - w\varkappa_{s\alpha_s}\alpha_{s\bar{\rho}}\right]^o \frac{A}{V_m + AL} \int_0^L \rho' dz \\
& \quad - (w\varkappa_{s\alpha_s}\alpha_{s\bar{T}})^o \left(\frac{A\int_0^L (dT/d\rho)\rho' dz}{V_m\rho_m + A\int_0^L \rho dz} \right. \\
& \quad \left. - \frac{T_m\rho_m V_m + A\int_0^L T\rho dz}{\left[V_m\rho_m + A\int_0^L \rho dz\right]^2} A\int_0^L \rho' dz \right) \Bigg\}
\end{aligned}
\tag{3.1}
$$

$$
\partial c_i{}'/\partial t = -\lambda_i c_i{}' + (\lambda_i c_i/n)^o\, n' \tag{3.2i}
$$

$$
\partial x'/\partial t = (b - \overline{v\sigma_x} x)^o n' - (\lambda_x + n\,\overline{v\sigma_x})^o x' \tag{3.3}
$$

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Equations (2.4) and (2.5) give

$$V\, \partial\rho_m'/\partial t = A[g_o' - g'(0)]$$

$$V\, (h_m + h_{m\rho_m}\rho_m)^o \partial\rho_m'/\partial t$$

$$= \int q_n{}^o n' d\tau - [g(0)h_{m\rho_m}]^o \rho_m'$$

$$+ A[h_o g_o' - h_m{}^o g'(0)]$$

which may be combined to eliminate g_o' thus obtaining

$$\partial\rho_m'/\partial t = V^{-1}[(h_m - h_o + \rho_m h_{m\rho_m})^o]^{-1}\{\int q_n{}^o n' d\tau \qquad (3.4)$$

$$- A[g(0)h_{m\rho_m}]^o \rho_m' + A(h_o - h_m)^o g'(0)\}$$

From Eq. (2.6)

$$\partial\rho'/\partial t = - \partial g'/\partial z$$

(3.5)

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From Eqs. (2.7) and (3.5)

$$\partial g'/\partial t = (Bg^m/\rho^2)^\circ \rho' + (g^2/\rho^2)^\circ \partial\rho'/\partial z$$

$$- (mB\rho^{-1}g^{m-1} + 2g\,\partial\rho^{-1}/\partial z)^\circ g' \qquad (3.6)$$

$$- (g/\rho)^\circ \partial g'/\partial z - \partial p'/\partial z$$

From Eqs. (2.8) and (3.5)

$$(\rho h_p - 1)^\circ \partial p'/\partial t$$

$$= [J_T T_\rho /A - (g/\rho^2)\partial p/\partial z]^\circ \rho'$$

$$- g^\circ(\partial/\partial z)(h_\rho^\circ \rho') \qquad (3.7)$$

$$+ (\rho^{-1}\partial p/\partial z + J_g/A - \partial h/\partial z)^\circ g'$$

$$+ (\rho h_\rho)^\circ \partial g'/\partial z + (J_\theta/A)^\circ \theta' ,$$

where terms on the right containing p' as a factor have been omitted after examination disclosed their unimportance. The new symbol J is defined as

$$J(g,\theta,T) = (\theta - T)H(g, \theta + T) \qquad (3.8)$$

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From Eq. (2.13) there results*

$$c\,\partial\theta'/\partial t = q_1 n' + \int q_2 n' d\tau - (J_T T_\rho)^{\circ}\rho' - J_g{}^{\circ} g' - J_\theta{}^{\circ}\theta' \tag{3.9}$$

It is also necessary to linearize the boundary conditions as expressed in Eqs. (2.10) - (2.12). This yields

$$\rho_m{}' = \rho'(0) \tag{3.10}$$

$$0 = p'(0) + [g(0)/\rho(0)]^{\circ} g'(0) - 1/2\,[g^2(0)/\rho^2(0)]^{\circ} \rho'(0) \tag{3.11}$$

$$0 = p'(L) + [g(L)/\rho(L)]^{\circ} g'(L) - 1/2\,[g^2(L)/\rho^2(L)]^{\circ} \rho'(L) \tag{3.12}$$

In the above equations the notation $f_x = \partial f/\partial x$ has been used. e.g. $K_{f\alpha_f} \equiv \partial K_f/\partial\alpha_f$.

* In equation (3.9), the quantity n' in the product $q_1 n'$ is to be regarded as the radial value of $n'(r,z)$ and is thus a function of z only. Such a treatment is required for a managable calculation, and maintains the identical treatment of the many fuel tubes.

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Section 4. THE VARIATIONAL PRINCIPLE AND THE ADJOINT EQUATIONS

The linearized partial differential equations derived above can be attacked by a study of the normal modes. It is to be remembered that a normal mode is a solution of the equations of motion, having the correct boundary conditions, for which all dependent variables (n, c_1, x, ρ , etc.), say f, have the time dependence $e^{\lambda t}$. For such a normal mode, the operator $\frac{\partial}{\partial t}$ is equivalent to multiplication by λ. Further, as long as coefficients in the equations do not explicitly contain the time, any solutions may be expanded in a series of normal modes with time independent coefficients.

If in equations (3.1)-(3.9), $\partial/\partial t$ is replaced by λ there results a set of equations which together with the boundary conditions will have a solution only for certain values of the reciprocal period λ. If the real part of λ is negative for all roots, the system is stable. In the work below, the emphasis centers on finding the values of λ.

A variational principle is a good way to obtain approximate eigenvalues in complex problems. Below, a variational principle for λ is constructed and the λ's evaluated by choosing suitable trial functions. (Unfortunately the variational principle of Rayleigh cannot be used, since the problem under

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consideration does not have self adjoint equations of motion).

Let N, C_i, X, R_m, R, P, G, Θ, be functions such that the expression

$$\lambda\ \Big(\int N n d\tau + \Sigma_i \int C_i c_i d\tau + \int X x d\tau + R_m \rho_m + \int R \rho d\tau$$

$$+ \int P p d\tau + \int G g d\tau + \int \Theta \theta d\tau\Big) \tag{4.1}$$

$$= \int N[K_f(1-\beta)\eta y - y - w\mathcal{L}_s]^o n' \, d\tau$$

$$+ \Sigma_i \lambda_i \int N K_f{}^o c_i' \, d\tau - \int N(y_x n)^o x' \, d\tau$$

$$+ \int N\{[K_{f\alpha_f}(1-\beta)\eta y n \alpha_{f\overline{\rho}} - w_{\overline{\rho}}\mathcal{L}_s - w\mathcal{L}_{s\alpha_s}\alpha_{s\overline{\rho}}]\overline{\rho}_{\rho_m}$$

$$- w\mathcal{L}_{s\alpha_s}\alpha_{s\overline{T}}\,\overline{T}_{\rho m}\}^o \, d\tau \ \rho_m'$$

$$+ \int N[K_{f\alpha_f}(1-\beta)\eta y n \alpha_{f\overline{\rho}} - w_{\overline{\rho}}\mathcal{L}_s - w\mathcal{L}_{s\alpha_s}\alpha_{s\overline{\rho}}]^o d\tau \int \overline{\rho}_\rho \rho' \, d\tau$$

$$- \int N(w\mathcal{L}_{s\alpha_s}\alpha_{s\overline{T}})^o d\tau \int \overline{T}_\rho \rho' \, d\tau$$

$$+ \Sigma_i(\lambda_i c_i/n)^o \int C_i n' d\tau - \Sigma_i \lambda_i \int C_i c_i' \, d\tau$$

$$+ \int X(b - \overline{v\sigma_x} x)^o n' d\tau - \int X(\lambda_x + n\overline{v\sigma_x})^o x' d\tau$$

$$+ [V(h_m - h_o + \rho_m h_{m\rho_m})^o]^{-1} R_m\{\int q_n{}^o n' d\tau + A(h_o - h_m)^o g'(0)$$

$$- A[g(0)h_{m\rho_m}]^o \rho_m'\}$$

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$$- \int R \partial g'/\partial z \, d\tau + \int G(Bg^m/\rho^2)^o \rho' d\tau + \int G(g^2/\rho^2)^o \partial\rho'/\partial z \, d\tau$$

$$- \int G(mB\rho^{-1}g^{m-1} + 2g\partial\rho^{-1}/\partial z)^o g' d\tau$$

$$- \int G(g/\rho)^o \partial g'/\partial z \, d\tau - \int G\partial p'/\partial z \, d\tau$$

$$+ \int P[(J_T T_\rho/A - g\rho^{-2}\partial p/\partial z)/(\rho h_p - 1)]^o \rho' \, d\tau$$

$$- \int P[g/(\rho h_p - 1)]^o \partial(h_\rho{}^o \rho')/\partial z \, d\tau$$

$$+ \int P[(\rho^{-1}\partial p/\partial z + J_g/A - \partial h/\partial z)/(\rho h_p - 1)]^o g' \, d\tau$$

$$+ \int P[\rho h_\rho/(\rho h_p - 1)]^o \, \partial g'/\partial z \, d\tau$$

$$+ \int P[A^{-1}J_\theta/(\rho h_p - 1)]^o \theta' \, d\tau + \int \Theta q_1/C \, n'^* d\tau$$

$$+ \int (\Theta/C) d\tau \int q_2 n' d\tau - \int \Theta(J_T T_\rho/C)^o \, \rho' \, d\tau$$

$$- \int \Theta(J_g{}^o/C) g' \, d\tau - \int \Theta(J_\theta{}^o/C)\theta' \, d\tau$$

has the property that the first variation, $\delta\lambda$, of λ vanishes for arbitrary variations, consistent withtthe boundary conditions, of the dependent variables n, c_i, etc. about their correct values and also for arbitrary variations consistent with the boundary conditions of the adjoint variables N, C_i, etc. about their correct values. This latter requirement is

*See footnote on Page 33.

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essentially automatic from the manner of construction of (4.1). Consideration of variations $\delta n'$ of n' etc. will give the required equations to be satisfied by the adjoint variables, together with the necessary boundary conditions. In the following evaluation of $\delta\lambda$ for variations $\delta n'$, $\delta c_i'$, etc., proper account of the boundary conditions (3.10) - (3.12) is included. Also, all derivatives of $\delta n'$, $\delta c_i'$, etc., are eliminated by partial integration. There results

$$\delta\lambda\left(\int N n' d\tau + \ldots\ldots + \int \Theta \theta' \, d\tau\right)$$

$$= \int \delta n \{-\lambda N + [(1-\beta)\eta y K_f - y - \varkappa_s w]^o N \tag{4.18}$$

$$+ \Sigma_i (\lambda_i c_i / n)^o \; C_i + (b - \overline{v\sigma_x} x)^o \; X$$

$$+ [V(h_m - h_o + \rho_m h_{m\rho_m})^o]^{-1} q_n{}^o R_m + (q_1/C) \; \Theta$$

$$+ q_2 \int (\Theta/C) d\tau \} + \Sigma_i \int \delta c_i (-\lambda C_i + \lambda_i K_f{}^o N - \lambda_i C_i) d\tau$$

$$+ \int \delta x [-\lambda X - (y_x n)^o N - (\lambda_x + n \; \overline{v\sigma_x})^o X] d\tau$$

$$+ \delta\rho_m \{-\lambda R_m + \int N \; [(K_{f\alpha_f}(1-\beta)\eta y n \alpha_{f\bar{\rho}} - w_{\bar{\rho}} \varkappa_s$$

$$- w \varkappa_{s\alpha_s} \alpha_{s\bar{\rho}}) \; \bar{\rho}_{\rho m} - w \varkappa_{s\alpha_s} \alpha_{s\bar{T}} \; \bar{T}_{\rho m}]^o \; d\tau$$

$$- [V(h_m - h_o + \rho_m h_{m\rho_m})^o]^{-1} \; A [g(0) h_{m\rho_m}]^o R_m \}$$

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$$+ \int \delta\rho \{ -\lambda R + \bar{\rho}_\rho \int N [K_{f\alpha_f}(1-\beta)\eta y n \alpha_{f\bar{\rho}} - w\bar{\rho}\mathscr{L}_s \qquad (4.18) \text{ cont'd.}$$

$$- w\mathscr{L}_{s\alpha_s}\alpha_{s\bar{\rho}}]^o d\tau - \bar{T}_\rho \int N (w\mathscr{L}_{s\alpha_s}\alpha_{s\bar{T}})^o \, d\tau$$

$$+ (Bg^m/\rho^2)^o G - \partial[(g^2/\rho^2)^o G]/\partial z$$

$$+ [(J_T T_\rho/A - g\rho^{-2}\partial p/\partial z)/(\rho h_p - 1)]^o \, P$$

$$+ h_\rho{}^o \, \partial[Pg^o/(\rho h_p - 1)^o]/\partial z - (J_T T_\rho/C)^o \Theta \} d\tau$$

$$+ A\delta\rho(L)[(g^2/\rho^2)^o G - Ph_\rho{}^o g^o/(\rho h_p - 1)^o]_{z=L}$$

$$- A\delta\rho(0)[(g^2/\rho^2)^o G - P\, h_\rho{}^o g^o/(\rho h_p - 1)^o]_{z=0}$$

$$+ \delta g(0)[V(h_m - h_o + \rho_m h_{m\rho_m})^o]^{-1} A(h_o - h_m)^o \, R_m$$

$$+ \int \delta g \{ -\lambda G + \partial R/\partial z - (mBg^{m-1}/\rho + 2g\partial\rho^{-1}/\partial z)^o \, G$$

$$+ \partial[(g/\rho)^o G]/\partial z - \partial[(\rho h_\rho)^o/(\rho h_p - 1)^o \, P]/\partial z$$

$$+ [(\rho^{-1}\partial p/\partial z + Jg/A - \partial h/\partial z)/(\rho h_p - 1)]^o \, P$$

$$- (Jg/c)^o \, \Theta \} d\tau$$

$$+ A\delta g(L)\{[\rho h_p/(\rho h_p - 1)]^o \, P - (g/\rho)^o \, G - R\}_{z=L}$$

$$- A\delta g(0)\{[\rho h_p/(\rho h_p - 1)]^o \, P - (g/\rho)^o \, G - R\}_{z=0}$$

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(4.18)
cont'd.

$$+ \int \delta p(-\lambda P + \partial G/\partial z)d\tau - A\delta p(L)G(L) + A\delta p(0)G(0)$$

$$+ \int \delta\theta\left\{-\lambda\Theta + \left[A^{-1}J_\theta/(\rho h_p - 1)\right]^o P - (J_\theta/c)^o\,\Theta\right\}d\tau$$

This mammoth expression gives directly the equations satisfied by the adjoint variables as follows from the remarks that $\delta\lambda$ must vanish for arbitrary δn, δc_i, $\delta\rho$, ... etc., subject to the boundary conditions. The boundary conditions restrict the variations $\delta\rho(o)$, $\delta p(o)$, $\delta g(o)$, $\delta\rho(L)$, $\delta p(L)$, $\delta g(L)$. In fact it is seen from (3.10) that $\delta\rho(o) = \delta\rho_m$. The equations for the adjoint functions are now written.

$$\lambda N = \left[(1-\beta)\eta y K_f - y - \alpha_s w\right]^o N + \textstyle\sum_i (\lambda_i c_i/n)^o C_i$$

$$+\left\{b - \overline{v\sigma_x}X\right\}^o X + \left[V^{-1}q_n/(h_m - h_o + \rho_m h_{m\rho_m})\right]^o R_m \tag{4.2}$$

$$+ (q_1/C)\,\Theta + (q_2/C)\int\Theta\,d\tau$$

$$\lambda C_i = \lambda_i K_f{}^o N - \lambda_i C_i \tag{4.3i}$$

$$\lambda X = -(y_x n)^o N - (\lambda_x + \overline{nv\sigma_x})^o X \tag{4.4}$$

The R_m equation (4.5) involves the boundary conditions

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and is discussed below.

$$\lambda R = -\bar{\rho}_\rho \int N\left[K_{f\alpha_f}(1-\beta)\eta\, y n \alpha_{f\bar{\rho}} - w_{\bar{\rho}} \mathcal{L}_s - w\mathcal{L}_{s\alpha_s}\alpha_{s\bar{\rho}}\right]^\circ d\tau - \bar{T}\rho \int N\left[w\mathcal{L}_{s\alpha_s}\alpha_{s\bar{T}}\right]^\circ d\tau$$
$$+ (Bg^m/\rho^2)^\circ G - \partial\left[(g^2/\rho^2)^\circ G\right]/\partial z$$
$$+ \left[(J_T T_\rho/A - g\rho^{-2}\partial p/\partial z)/(\rho h_p - 1)\right]^\circ P$$
$$+ h_\rho{}^\circ \partial\left\{\left[g/(\rho h_p - 1)\right]^\circ P\right\}/\partial z - (J_T T_\rho/C)^\circ\, \Theta \tag{4.6}$$

$$\lambda G = \partial R/\partial z - (mBg^{m-1}/\rho + 2g\partial\rho^{-1}/\partial z)^\circ\, G$$
$$+ \partial\left[(g/\rho)^\circ G\right]/\partial z + \left[(\rho^{-1}\partial p/\partial z + Jg/A - \partial h/\partial z)/(\rho h_p - 1)\right]^\circ P$$
$$- \partial\left\{\left[\rho h_\rho/(\rho h_p - 1)\right]^\circ P\right\}/\partial z - (Jg^\circ/C)\,\Theta \tag{4.7}$$

$$\lambda P = \partial G/\partial z \tag{4.8}$$

$$\lambda\Theta = A^{-1}\left[J_\theta/(\rho h_p - 1)\right]^\circ P - (J_\theta/C)^\circ\, \Theta \tag{4.9}$$

The remaining terms including those of (4.5) are boundary terms and result in the following type of equation.

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$$0 = \delta\rho_m' \{-\lambda R_m + \ldots\}_1 + \delta\rho'(0)\{\quad\}_2$$

$$+ \delta g'(0)\{\quad\}_3 + \delta p'(0)\{\quad\}_4 + \delta\rho'(L)\{\quad\}_5$$

$$+ \delta g'(L)\{\quad\}_6 + \delta p'(L)\{\quad\}_7$$

If all the δ's appearing here were independent, each $\{\quad\}$ would have to vanish and seven equations would result. Because of the three boundary conditions (3.10, 3.11, 3.12), however, only four equations result. The just mentioned boundary conditions give

$$\delta\rho'(0) = \delta\rho'_m$$

$$\delta p'(0) = 1/2\left[g^2(0)/\rho^2(0)\right]^o \delta\rho'_m - \left[g(0)/\rho(0)\right]^o \delta g'(0)$$

$$\delta p(L) = 1/2\left[g^2(L)/\rho^2(L)\right]^o \delta\rho'(L) - \left[g(L)/\rho(L)\right]^o \delta g'(L)$$

which when substituted in the preceding equation results in

$$0 = \delta\rho_m'\left\{\{\quad\}_1 + \{\quad\}_2 + \frac{1}{2}\left(\frac{g^2(0)}{\rho^2(0)}\right)\{\quad\}_4\right\}_a$$

$$+ \delta g'(0)\left\{\{\quad\}_3 - \left(\frac{g(0)}{\rho(0)}\right)^o\{\quad\}_4\right\}_b$$

$$+ \delta\rho'(L)\left\{\{\quad\}_5 + \frac{1}{2}\left(\frac{g^2(L)}{\rho^2(L)}\right)^o\{\quad\}_7\right\}_c$$

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$$+ \ \delta g'(L)\left\{\{\ \}_6 - \left(\frac{g(L)}{\rho(L)}\right)^{o}\{\ \}_7\right\}_d$$

Each of these $\{\ \}_{a,\ b,\ c,\ d}$ may now be set equal to zero. The equation $\{\ \}_a = 0$ is written as equation (4.5). The other three give the remaining adjoint boundary conditions.

$$\begin{aligned}\lambda R_m = & \int N\left\{\left[K_{f\alpha_f}(1-\beta)\eta y n \alpha_{f\bar{\rho}} - w_{\bar{\rho}}\alpha_s - w\alpha_{s\alpha_s}\alpha_{s\bar{\rho}}\right]\bar{\rho}_{\rho m}\right. \\ & \left. - w\alpha_{s\alpha_s}\alpha_{s\bar{T}}\bar{T}_{\rho m}\right\}^{o} d\tau \qquad (4.5) \\ & - V^{-1}\left[(h_m - h_o + \rho_m h_{m\rho_m})^{o}\right]^{-1} A\left[g(0)h_{m\rho_m}\right]^{o} R_m \\ & + A\left\{\left[h_\rho g/(\rho h_p - 1)\right]^{o} P - \frac{1}{2}(g^2/\rho^2)^{o} G\right\}_{z=0}\end{aligned}$$

$$\begin{aligned}0 = & \left\{V^{-1}\left[(h_o - h_m)/(h_m - h_o + \rho_m h_{m\rho_m})\right]^{o} R_m\right. \qquad (4.10) \\ & \left. + R - \left[\rho h_\rho/(\rho h_p - 1)\right]^{o} P\right\}_{z=0}\end{aligned}$$

$$0 = \left\{\frac{1}{2}(g^2/\rho^2)^{o} G - \left[h_\rho g/(\rho h_p - 1)\right]^{o} P\right\}_{z=L} \qquad (4.11)$$

$$0 = \left\{\left[\rho h_\rho/(\rho h_p - 1)\right]^{o} P - R\right\}_{z=L} \qquad (4.12)$$

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Section 5. Trial Functions and the Matrix

In the preceding section it has been shown how to select adjoint variables such that (4.1) is a stationary expression for λ. This means that the error in λ as determined from (4.1) varies as the product of the errors in the dependent variables and the adjoint variables. Thus one method of obtaining an estimate of λ is to guess at values for the dependent and adjoint variables. The method of making such a guess as used in this work is discussed in this section.

If the reactor is imagined to operate slightly away from its equilibrium configuration because of some sort of weak disturbance, the power is expected to change in magnitude but hardly at all in shape. That is, it is expected that the neutron density is simply multiplied by a position independent factor. This behavior of a reactor follows from the relatively weak coupling of power and reactivity, considered together with the strong damping of the other possible space distributions of neutrons.

The above considerations indicate that the neutron flux variation, n', will have the same form as the flux, n^o, i.e., $n' = c(t)n^o$ for some space independent coefficient $c(t)$. Similarly $N = C(t)N^o$ where N^o is the adjoint neutron flux. The

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adjoint flux has not been discussed in the above, but clearly it exists since the neutron equations are linear in the variables n and c_i. Thus the equation for n^o from (2.1) and (2.2) is

$$0 = (K_f \eta y)^o n^o - (y + w \alpha_s)^o n^o$$

whence

$$0 = (\eta y K_f)^o N^o - (y + w \alpha_s)^o N^o \qquad (5.1)$$

This may be compared with (4.2) and (4.3) in the case where $\lambda = 0$ and all variables but N and C_i are set equal to zero. A solution of (5.1) is given by

$$\alpha_s^o N^o = y^o n^o \qquad (5.2)$$

The remaining trial functions may now be obtained as follows. Ignore the $\lambda n'$ and the λN equations and use $n' = n^o$, $N = N^o$ in the remaining equations to find the dependent and adjoint variables. Once all the variables are determined in such a manner, equation (4.1) results in a value for λ.

This procedure may be interpreted as follows. For slow changes in N and n the other variables will at any instant of time have nearly the equilibrium values they would attain if n' and N suddenly became constant. It is these equilibrium values that are computed. Thus the pro-

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cedure outlined should give the longest reactor period.

This sort of approach is analogous to the well known Rayleigh variational principle. It can be modified in a manner similar to that applied by Ritz to the Rayleigh principle. That is, trial functions are used which depend on a set of parameters and then the resultant λ, calculated from (4.1) is made stationary with respect to the parameters. This procedure gives as many roots λ as there are parameters, each root corresponding to a normal mode. If the form of the trial functions is wisely chosen then the resulting normal modes can adequately represent the time behavior of the system.

One caution is necessary in the formation of a parameter-dependent set of trial functions; namely, the boundary conditions must be satisfied for all values of the parameters. This problem enters especially in the treatment of the hydrodynamics of the system.

In order to take separate account (in the trial functions) of the effects of heating of the moderating water and the heating of the fuel tube, the following two sets of functions are defined, and are called Mode I solutions and Mode II solutions of the linearized equations.

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Mode I (heating of the fuel tube only)

$\rho_m^{(1)}$, $\rho^{(1)}$, $g^{(1)}$, $\theta^{(1)}$ satisfy equations (3.4) to (3.12) with $n' = n^o$ equation (3.9) and $n' = 0$ in equation (3.4). Similarly $R_m^{(1)}$, $R^{(1)}$, $P^{(1)}$, $G^{(1)}$, $\Theta^{(1)}$ satisfy Eqs. (4.5) to (4.12) with $N=N^o$ in Eq. (4.6) and $N=0$ in Eq. (4.5).

Mode II (heating of moderator water only)

The variables are indicated with a superscript as $\rho^{(2)}$, and are defined as are the Mode I functions except that

$$n' = 0 \text{ in Eq. (3.9)} \quad ; \quad n' = n^o \text{ in Eq. (3.4)}$$

$$N = 0 \text{ in Eq. (4.6)} \qquad N = N^o \text{ in Eq. (4.5)}$$

The trial functions are given in Appendix V.

In accordance with these remarks, the trial functions are taken to be:

$$
\begin{array}{ll}
n' = u_1 n^o & N' = U_1 N^o \\
c_1' = u_2 c_1^o & C_1' = U_2 K_f^o N^o \\
c_2' = u_3 c_2^o & C_2' = U_3 K_f^o N^o \\
x' = u_4 x^o & X' = U_4 N^o \; * \\
\rho_m' = u_5 \rho_m^{(1)} + u_6 \rho_m^{(2)} & R_m = U_5 R_m^{(1)} + U_6 R_m^{(2)} \\
\rho' = u_5 \rho^{(1)} + u_6 \rho^{(2)} & R = U_5 R^{(1)} + U^6 R^{(2)} \\
g' = u_5 g^{(1)} + u_6 g^{(2)} & G = U_5 G^{(1)} + U^6 G^{(2)} \\
p' = u_5 p^{(1)} + u_6 p^{(2)} & P = U_5 P^{(1)} + U^6 P^{(2)} \\
\theta' = u_7 \theta^{(1)} + u_8 \theta^{(2)} & \Theta = U_7 \Theta^{(1)} + U_8 \Theta^{(2)}
\end{array}
$$

*This uses the facts that $\frac{\partial y}{\partial x} = \overline{v\sigma}_x$ and that $n^o \; \overline{v\sigma}_x \gg \lambda_x$.

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When these trial functions are inserted into Equation (4.1) there results an expression of the form

$$\lambda\Sigma_{i,j}\, m_{ij}U_i u_j = \Sigma_{i,j} M_{ij}U_i u_j$$

The condition that λ be stationary for variation of the U_i leads to

$$\lambda\Sigma_j m_{ij}u_j = \Sigma_j M_{ij}u_j \tag{5.3}$$

In order that this equation be solvable for the u_j, the determinant equation

$$|M - \lambda m| = 0 \tag{5.4}$$

must be satisfied. Equation (5.4) gives eight roots λ. Equation (5.3) then determines the ratios among the u_j for each normal mode in termsof the λ for that mode. Results concerning the matrices M and m are given in Appendix VI.

Finding the roots of Equation (5.4) is a standard but not trivial problem. Several approaches were used including straight forward expansion of the determinant.

Appendix I presents the values of the roots along with results of reactor response to reactivity changes as computed by use of these roots and of their corresponding normal modes. Also given in Appendix I are further results based on a simplified reactor model discussed in the next section.

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Section 6. SIMPLER MODELS

The reactor dynamics calculations discussed above are tedious and time consuming to perform. It would be desirable to use the results of this analysis to construct a simplified (and necessarily crude) model to be used to extend the calculations of the transient behavior of the reactor to include consideration of step changes in flow rate, in water temperature at the inlet to the moderator region, in water temperature at the inlet to the fuel elements, in the moderator-coolant apportioning valve setting. The model can also be used to investigate the effect of boundary conditions different from those employed above. The remainder of this section will be devoted to devising such a useful model.

The reactor is considered to be comprised of six systems: (1) fission, (2) and (3) delayed neutron emitters, (4) moderating water, (5) cooling water, and (6) heat transfer steel. Since the model is to be used for the calculation of dynamic effects over short periods of time, the long time effects such as xenon burnout will be neglected.

The complex fuel element system is treated as a system with only two degrees of freedom, the variables being the maximum fluid temperature and the corresponding fluid temperature.

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Section 6.1 THE EQUATIONS OF MOTION

It is convenient to take as variables n, c_1, c_2, $\overline{T}$, and ρ_m, defined previously, plus

T_I = maximum wall (steel) temperature

and

T = fluid temperature at position where the wall temperature is a maximum.

The equations for the neutron density and concentrations of delayed neutron emitters can be written:

$$\partial n/\partial t = \left[R(\rho_m,t)(1-\beta)-1\right] n/\ell + \Sigma_i \lambda_i c_i \tag{6.1}$$

$$\partial c_i/\partial t = -\lambda_i c_i + \beta_i n/\ell \tag{6.2i}$$

where ℓ is the neutron lifetime against absorption and R is the reactivity, a function of ρ_m and T. Its equilibrium value is unity.

Linearization of equation 6.1 yields

$$\partial n'/\partial t = -\beta n'/\ell + n^o R'/\ell + n^o \Sigma_i \lambda_i c_i'$$

or

$$\frac{1}{n^o}\frac{\partial n'}{\partial t} = \frac{1}{\ell}\left[-\beta\frac{n'}{n^o} + R' + \Sigma_i \beta_i \frac{c_i'}{c_i^o}\right] \tag{6.3}$$

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where $R'(1-\beta)$ has been approximated by R' and $\lambda_i c_i^{o}$ has been replaced by $\beta_i n^{o}/\ell$. Equations 6.2i are linearized exactly as before to obtain:

$$\frac{1}{\lambda_i}\frac{\partial}{\partial t}\frac{c_i'}{c_i^{o}} = -\frac{c_i'}{c_i^{o}} + \frac{n'}{n^{o}} \tag{6.4i}$$

The change in reactivity can be written

$$R' = (\partial R/\partial \rho_m)\rho_m' + (\partial R/\partial T)\, T'$$

It is convenient to express R' in units of β (i.e. dollars) and to define dimensionless quantities

$$A_{\rho_m} = \frac{(\rho_m')_1}{\beta}\;\frac{\partial R}{\partial \rho_m}$$

and

$$A_T = \frac{(T')_1}{\beta}\;\frac{\partial R}{\partial T}$$

Here $(x')_1$ means the equilibrium value of x' for a unit fractional power change ($n' = n^{o}$), all other variables in equations 6.1 - 6.4i plus the flow rate being kept constant. In terms of these quantities,

$$R'/\beta \text{ (dollars)} = A_{\rho_m}\,\rho_m'/(\rho_m')_1 + A_T T'/(T')_1$$

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The quantity A_x has the following meaning. If the power is increased to a new steady value and all variables except x are held constant, then A_x times the fractional power change is the change in reactivity expressed in dollars due to system x. Equation 6.3 can now be written

$$\frac{\ell}{\beta}\frac{d}{dt}\frac{n'}{n^o} + \frac{n'}{n^o} = A\rho_m \frac{\rho_m'}{(\rho_m')_1} + A_T \frac{T'}{(T')_1} + \sum_i \frac{\beta_i}{\beta}\frac{c_i'}{c_i^o} \qquad (6.5)$$

The determination of the quantities ℓ/β, $A\rho_m$, and A_T will be discussed in section 6.2.

The heat balance equation in the moderating region can be written

$$M dh_m/dt = q_m - Ag(h_m - h_o) \qquad (6.6)$$

where M is the mass of moderating water and q_m the power supplied to it. When this equation is linearized it yields

$$\tau_m \frac{d}{dt}\frac{\rho_m'}{(\rho_m')_1} + \frac{\rho_m'}{(\rho_m')_1} = \frac{n'}{n^o} - \frac{g'}{g^o} + \frac{\rho_m'^{\,in}}{(\rho_m')_1} \qquad (6.7)$$

Where we have used the relation

$$h'_m = \frac{\partial h_m}{\partial \rho_m}\,\rho_m'$$

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and have placed* $q_m'/q_m^o = n'/n^o$, $\rho_m'^{in}$ is the change in water density at the inlet to the moderator region. If either the neutron density, the flow rate, or inlet water density (or all three of them) were suddenly changed to new steady values, ρ_m' would approach some equilibrium value. After a time τ_m, it would differ from this equilibrium value by less than a factor 1/e. This time is taken here to be the transit time of fluid through the moderating region.

The equation relating the wall temperature T_I to the fluid temperature T, the flow rate g, and the power supplied to the walls q_I is:

$$C_I \frac{d}{dt} T_I = q_I - gF(T_I + T) \left[T_I - T\right]$$

Here C_I is the heat capacity of the iron walls and, as before, gF is the heat transfer coefficient and is a function of $T_I + T$. This equation is somewhat complicated and we shall carry out its linearization in detail. As a first step we obtain:

* In this simple model no attempt is made to distinguish between prompt and delayed moderator heating. It is all taken to be prompt.

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$$C_I \frac{d}{dt} T_I' = q_I' - g^o F^o \left[T_I^o - T^o\right] \left\{\frac{g'}{g^o} + \frac{T_I' - T'}{T_I^o - T^o} + \frac{F'}{F^o}\right\}$$

Now

$$F' = \frac{\partial F}{\partial T_I} T_I' + \frac{\partial F}{\partial T} T' = \frac{dF}{d(T_I + T)} (T_I' + T')$$

We define

$$-\frac{1}{F^o} \frac{dF}{d(T + T_I)}$$

to be

$$\frac{\alpha}{T_I^o - T^o}$$

Thenthe above equation becomes, upon dividing by $g^o F^o$,

$$\frac{C_I}{g^o F^o} \frac{d}{dt} T_I' = (T_I^o - T^o)(q_I'/q_I^o - g'/g^o) - T_I'(1-\alpha) + T'(1+\alpha)$$

Placing $q_I'/q_I^o = n'/n^o$ one obtains the equation

$$\frac{C_I}{g^o F^o} \frac{d}{dt} T_I' + T_I' = \frac{T_I^o - T^o}{1-\alpha} \left(\frac{n'}{n^o} - \frac{g'}{g^o}\right) + T' \frac{1+\alpha}{1-\alpha} \qquad (6.8)$$

From the steady state data $C_I/g^o F^o$ is found to equal .09 sec, and $T_I^o - T^o = 450^o$ F. α is determined from some

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of the known results of the calculation in the previous sections. If a 1% increase in power into the fuel tubes ($n'/n^o = .01$) at constant flow rate ($g'/g = 0$) occurs, then equations 3.5 - 3.9 show that the maximum iron temperature is changed by 10° F and the fluid temperature at the place where T_I' is a maximum is changed by 2.5° F. When these values are used in equation 6.8, α is found to be very nearly 1/4. Equation 6.8 then becomes

$$.012 \frac{d}{dt} T_I' + T_I' = 584\left(\frac{n'}{n^o} - \frac{g'}{g^o}\right) + \frac{5}{3} T' \tag{6.9}$$

It will be noticed that all of the linearized equations are of the form

$$\tau dx/dt + x = \text{forcing terms}$$

The forcing terms contain those variables that directly influence the value of x. The equation for T' will then be of the form*

$$\tau dT'/dT + T' = \epsilon T_I' + \gamma\, T_{in}'$$

*g' does not enter this equation for the following reason. If the flow rate is decreased by, say 1% the fluid will remain in the fuel element and be heated for 1% longer time. However, the heat transfer coefficient is directly proportional to g and will be reduced 1%. The result is that the fluid temperature will be unchanged.

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where T_{in} is the inlet temperature to the fuel element. From the results mentioned above for T_I' and T', ϵ is readily found to equal $\frac{2.50}{10} = 1/4$. The time constant τ is arbitrarily taken to equal the transit time of the fluid through the fuel element, viz., 0.1 sec. The constants α and ϵ have been unambiguously assigned values so that this simple model gives the same equilibrium values of T' and T_I', following a power change, as the detailed calculation. However, γ can be chosen in at least two ways. It can be chosen so that a change in inlet water temperature T_{in} produces the correct change in T, or it can be selected so that the effect of a change in T_{in} on the reactivity is given properly. This latter choice was made and it will be shown that this leads to placing γ equal to 5/8.

The last two equations can be written in terms of the equilibrium values $(T')_1$, $(T_I')_1$ following a unit fractional power change with all quantities but these two held constant.

$$0.12 \frac{d}{dt} \frac{T_I'}{(T_I')_1} + \frac{T_I'}{(T_I')_1} = \frac{7}{12}\left(\frac{n'}{n^o} - \frac{g'}{g^o}\right) + \frac{5}{12} \frac{T'}{(T')_1} \qquad (6.10)$$

$$0.10 \frac{d}{dt} \frac{T'}{(T')_1} + \frac{T'}{(T')_1} = \frac{T_I'}{(T_I')_1} + \frac{5}{8} \frac{T_{in}'}{(T')_1} \qquad (6.11)$$

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Section 6.2 THE DETERMINATION OF NEUTRON LIFETIME AND REACTIVITY PARAMETERS

The variational technique described previously was designed specifically to yield the inverse periods $\frac{1}{n'}\frac{dn'}{dt} = \lambda_i$, with which the pile moves from its equilibrium position. Equation 6.1 shows that the reactivity R, and hence the coefficient A_{ρ_m} and A_T, is related to the quantity $1/n\ dn'/dt$. In linearized form this is $1/n^o\ dn'/dt$ and will be called $\varkappa$. The effects of the fuel element and moderator systems on $\varkappa$ can be found more directly than their effects on the reactivity. Equation 6.5 can be used to relate changes in $\varkappa$ to the coefficients A_T and A_{ρ_m}. For the purpose of this calculation delayed neutrons are unimportant and all neutrons will be taken as prompt by placing β and the β_i equal to zero. In the limiting case equation 6.5 takes the form:*

$$\frac{1}{n^o}\frac{dn'}{dt} = \varkappa = \frac{\beta}{\ell}\left\{A_{\rho_m}\frac{\rho_m'}{(\rho_m')_1} + A_T\frac{T'}{(T')_1}\right\} \qquad (6.12)$$

* The coefficients A_{ρ_m} and A_T were defined containing a factor of $1/\beta$, hence β times them is non-zero in the limiting case where β approaches zero.

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If it is recalled that the quantity λ in equation 5.3 is equivalent to the operator $\frac{d}{dt}$, then this equation can be written

$$\sum_j m_{ij} \frac{d}{dt} \mu_j = \sum_j M_{ij} \mu_j$$

From the definition of $u_1 = \frac{n'}{n^o}$ and from examination of the matrix m in Appendix VI, it is clear upon comparison with equation 6.3 that matrix element M_{11} is equal to $- \beta/\ell$. Appendix VI then shows that:

$$\beta/\ell = 161.5 \text{ sec}^{-1}.$$

Hence if the effect of the moderator system on $\varkappa$ is known following a unit fractional power change $(\rho_m' = (\rho_m')_1)$, then the knowledge of β/ℓ yields A_{ρ_m}. A_T is determined similarly from a knowledge of the fuel element effect on $\varkappa$. The determination of these effects will now be carried out.

Neglecting delayed neutrons, which are unimportant for their calculation, the equation for the neutron density becomes

$$\frac{\partial n}{\partial t} = \varkappa n = K_f \eta y n - (y + w\varkappa_s) n$$

Multiplying this equation by the adjoint neutron function N

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and integrating over all space yields an equation for $\varkappa$ which is invariant to changes in n and N. This is

$$\varkappa \int N n \, d\tau = \int N \left\{ K_f \eta y n - (y + w \alpha_s) n \right\} d\tau$$

It is desired to find the effect of the moderating and cooling water properties on $\varkappa$. This means that changes in K_f, w and α_s will be examined. The parameters that these quantities depend upon were given in Section 3.

The value of $\varkappa$ for the steady state conditions is, of course, zero. The change in $\varkappa$ for changes in water properties is then given by

$$\delta\varkappa \int N n \, d\tau = \varkappa \int N n \, d\tau = \int N \left\{ \delta K_f \eta y n - (\delta w \alpha_s + w \delta \alpha_s) n \right\} d\tau$$

$$= \bar{\rho}' \int N^{o} \left\{ \frac{dK_f}{da_f} \frac{da_f}{d\bar{\rho}} \eta y n - \frac{dw}{d\bar{\rho}} \alpha_s n + w \frac{d\alpha_s}{da_s} \frac{da_s}{d\bar{\rho}} n \right\}^{o} d\tau$$

$$+ \bar{T}' \int N^{o} \left\{ w \frac{d\alpha_s}{da_s} \frac{da_s}{d\bar{T}} n \right\}^{o} d\tau$$

The first term gives the effect of the average density on the inverse pile period, and the second the effect of the average temperature. The integrals can be evaluated from knowledge of the steady state functions and yield for $\varkappa$

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$$\varkappa = 5080 \frac{\bar{\rho}'}{\bar{\rho}^o} + 240 \frac{\bar{T}'}{\bar{T}}$$

$\bar{\rho}$ was defined in Section 2 as

$$\bar{\rho} = \frac{\rho_m V_m + A\int_0^L \rho \, dz}{V_m + AL}$$

$\bar{\rho}'/\bar{\rho}^o$ can be broken up into two parts, a moderator and a fuel element part:

$$\frac{\bar{\rho}'}{\bar{\rho}^o} = \frac{V_m \rho_m'}{\rho_m^o V_m + A\int_0^L \rho^o dz} + \frac{A\int_0^L \rho' \, dz}{\rho_m^o V_m + A\int_0^L \rho^o dz}$$

For a unit fractional power change ($n' = n^o$) and a constant flow rate ($g' = 0$) equation 3.4 can be solved for the equilibrium value of ρ_m' $[$i.e. $(\rho_m')_1]$. Similarly equations 3.7 and 3.9 can be solved for the equilibrium value of $\rho'(z)$ for constant flow rate ($g' = 0$) and the boundary condition $\rho'(0) = 0$, for a unit fractional power change. In this way the contribution of the moderator and coolant water density to $\varkappa$ for $n' = n^o$ can be obtained.

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Using equation 6.12 and the value for β/ℓ we obtain

$$A_{\rho_m} = - \frac{436}{161.5} = - 2.7 \text{ dollars/unit fractional power change}$$

$$A_T = - \frac{129}{161.5} = - 0.8 \quad " \quad " \quad " \quad " \quad "$$

From a knowledge of ρ_m' and $\rho'(z)$, $\overline{T}'$ can be computed. In the same manner as above the reactivity effect due solely to the water temperature can be calculated. This yields a value of +0.12 \$/unit fract. power change. This gives a preliminary table of reactivity coefficients* as follows:

PRELIMINARY REACTIVITY COEFFICIENTS

TABLE 6.1

Source	Reactivity Coefficient, (\$)
Moderator density	-2.7
Coolant density	-0.8
Average temperature	+0.12

* For the slow variations discussed in ORNL-1177 it is necessary to know the xenon power coefficient of reactivity. y is a function of the atomic density of xenon atoms x but ηy is not. Hence the effect of Xe on $\varkappa$ is determined from.

$$\varkappa \int N n \, d\tau = - \int N^o \frac{dy}{dx} x' n^o d\tau = - \frac{x'}{x^o} \int N^o \frac{dy}{dx} x^o n^o \, d\tau$$

Its reactivity coefficient is found to be +3.8 dollars per unit fractional power change.

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The coolant density coefficient was calculated assuming that the coolant water affects the reactivity only through the average water density $\bar{\rho}$. Actually the fuel elements are clustered together at the center of the core and would be expected to have a greater effect. For this reason the coolant reactivity coefficient is arbitrarily altered from -0.8 to -1.2 dollars. The average temperature coefficient was computed assuming $\overline{v\sigma}_x$ was independent of temperature. Reference to figure 13 in ORNL-1177 shows that at worst this effect could increase the temperature coefficient from 0.12 to 0.20. To be conservative the latter value will be used. The final power coefficient of reactivity are then:

POWER COEFFICIENTS OF REACTIVITY

TABLE 6.2

Source	Reactivity Coefficient, ($)
Moderator density	-2.7
Coolant density	-1.2
Average temperature	+0.20

The temperature coefficient is not divided into a fuel tube and moderator portion because its effect is small and nearly all of it is due to the moderator water. This moderator temperature effect is treated separately here from the

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density effect because it is unstabilizing and it could conceivably act sufficiently faster than the stabilizing density effect to cause trouble. This would be the case, for example, if for some reason it was exceedingly difficult to reduce the amount of water in the reactor in a short time. Then an increase in power would cause an increase in temperature and a build up of pressure which might not be followed by a stabilizing decrease in density for a time long enough to allow the pile power to reach a dangerous level. The pile period which would result if there were only delayed neutrons to hold down the tendency of the power to run away because of the unstabilizing temperature effect has been calculated to be about 33 seconds. It is estimated that the density change will follow the temperature change within tens of milliseconds. Therefore, in what follows no distinction will be made between the moderator density and temperature effects and a combined coefficient of -2.5 dollars per unit fractional power change will be used.

Finally the parameter γ in the equation for T' must be estimated. γ determines the effect of changing the water temperature at the inlet to the fuel element. If at constant power ($n' = 0$) and constant flow rate ($g' = 0$) the temperature at the inlet to the fuel tube is changed, the enthalpy along

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the entire length of the fuel tube will be changed by a fixed amount, independent of position. By use of equations 3.7 and 3.9 the equilibrium value of $\rho(z)$ has been calculated for a temperature increase of $1^{\circ}F$ at the inlet to the fuel tube, subject to the above conditions. This temperature causes a 9.25×10^{-3} percent decrease in the average density $\bar{\rho}$. Hence the reactivity effect is calculated to be:

$$\frac{\chi}{161.5} = - \frac{5080(9.25\times10^{-5})}{161.5} = -0.003 \frac{\text{dollars}}{^{\circ}F}$$

For the same reason as before we increase this fuel tube coefficient and round it off to $-.005$ dollars/$^{\circ}$F. From the equations for T_I' and T' we find the equilibrium value of T' for a one degree increase in T_{in} to be (cf. Section 6.1)

$$T' = \frac{1}{4} T_I' + \gamma = \frac{5}{12} T'$$

$$T' = \frac{12\gamma}{7}$$

The change in reactivity due to this change in T' is

$$A_T \frac{12\gamma}{7(T')_1} = \frac{12\gamma}{7} \frac{(-1.20)}{250} = -0.008\gamma = -0.005 \text{ dollars}$$

$$\left[(T')_1 = 250^{\circ}F\right]$$

Therefore γ is approximately 5/8.

The equations of motion are summarized below.

$$0.0063 \frac{d}{dt}\frac{n'}{n^{o}} + \frac{n'}{n^{o}} = -2.5 \frac{\rho_m'}{(\rho_m')_1} - 1.20 \frac{T'}{(T')_1} \qquad (6.5a)$$

$$+ 0.465 \frac{C_1'}{C_1^{o}} + 0.535 \frac{C_2'}{C_2^{o}} + \delta R$$

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Where δR is the reactivity (in dollars) externally imposed on the reactor (e.g. via a control rod).

$$\frac{1}{\lambda_i}\frac{d}{dt}\frac{C_i'}{C_i^o} + \frac{C_i'}{C_i^o} = \frac{n'}{n^o} \qquad (6.4i)$$

$$2.3\,\frac{d}{dt}\frac{\rho_m'}{(\rho_m')_1} + \frac{\rho_m'}{(\rho_m')_1} = \frac{n'}{n^o} - \frac{g'}{g^o} + \frac{\rho_m'^{\,in}}{(\rho_m')_1} \qquad (6.7)$$

Recalling that $(T_I')_1 = 1000^o$ F and $(T')_1 = 250^o$ F:

$$0.12\,\frac{d}{dt}\frac{T_I'}{(T_I')_1} + \frac{T_I'}{(T_I')_1} = \frac{7}{12}\frac{n'}{n^o} - \frac{g'}{g^o} + \frac{5}{12}\frac{T'}{(T')_1} \qquad (6.10)$$

$$0.10\,\frac{d}{dt}\frac{T'}{(T')_1} + \frac{T'}{(T')_1} = \frac{T_I'}{(T_I')_1} + \frac{5}{8}\frac{T_{in}'}{(T')_1} \qquad (6.11)$$

τ_m is taken to be the time required to change completely the water in the moderator tank once and is about 2.3 seconds.

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Section 6.3 FLOW CHOKING

The equilibrium values of ρ_m' and T', and hence their effect on pile reactivity, depend upon the equilibrium value of g'/g^o. A particular case of interest is when g'/g^o depends upon the fluid temperature T'. This occurs, for example, when a constant pressure difference is maintained across the fuel element or when the fluid exits into a device whose flow resistance increases with temperature. Then an increase in exit temperature will cause a decrease in flow rate. To approximate this condition g'/g^o will be taken proportional to $T'/(T')_1$. It is convenient to define a flow factor c by the following relation between the equilibrium values of g' and n'.

$$g'/g^o = - c\, n'/n^o \tag{6.13}$$

When T_{in}' is zero, the equilibrium value of $T'/(T')_1$ is $(1+c)\frac{n'}{n^o}$ (cf. equ. 6.10 and 6.11). During the transient then g'/g^o will be taken equal to

$$g'/g^o = - \frac{c}{1+c} \quad \frac{T'}{(T')_1} \tag{6.14}$$

In most cases T' follows changes in n much more rapidly than

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does ρ_m'. Consequently little error is introduced by using 6.13 in the equation for ρ_m instead of 6.14. Equations 6.7, 6.10 now become

$$2.3 \frac{d}{dt} \frac{\rho_m'}{(\rho_m')_1} + \frac{\rho_m'}{(\rho_m')_1} = \frac{n'}{n^o}(1+c) + \frac{\rho_m'}{(\rho_m')_1} \qquad (6.15a)$$

$$0.12 \frac{d}{dt} \frac{T_I'}{(T_I')_1} + \frac{T_I'}{(T_I')_1} = \frac{7}{12}\frac{n'}{n^o} + \frac{5}{12}\frac{T'}{(T')_1}\left[1+1.4\frac{c}{1+c}\right] \qquad (6.15b)$$

The detailed calculation which was carried out with constant pressure boundary conditions exhibited a flow factor c of about 5. Figure 2 shows the results the calculation of the power change following a step change in reactivity using the simple model* for flow factors of 0 and 5. On this same figure the result of the detailed calculation also appears. The effect of the flow variation is quite apparent. It is also obvious that the simple model underestimates the time required for the transient to die away. This probably indicates that either or both of the response times used in the T_I' and T' equations are too short.

* Only one delay group was employed in this calculation so the results are only meaningful for times up to about two seconds or so.

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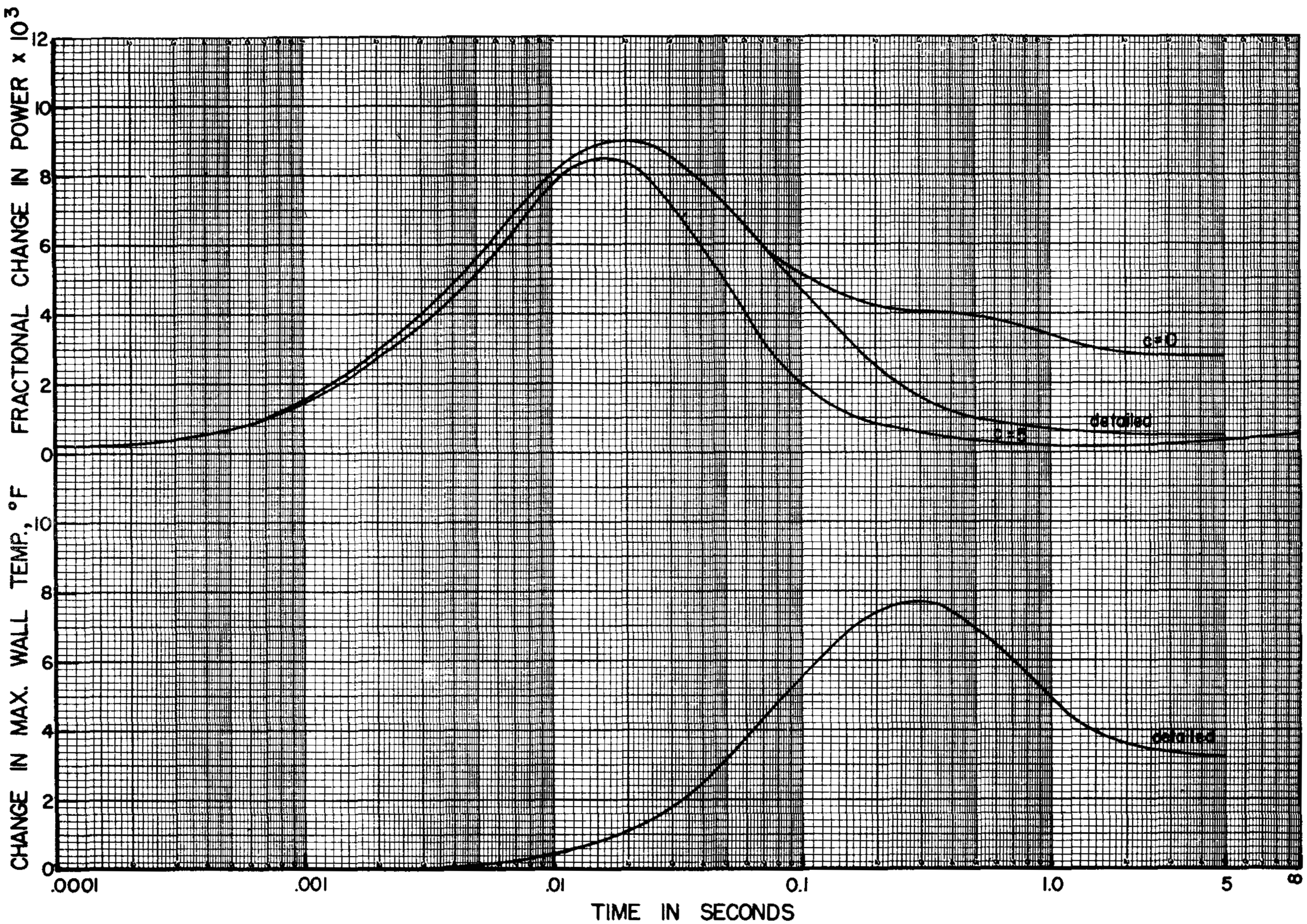

Responses to a One Cent Step Change in Reactivity. The curves marked "detailed" are the results of the variational calculation. The others are results obtained from the simple model of Section 6 for two values of the "choking factor" c.

FIG. 2
12-TR-2

[redacted]

Two other calculations using the simple model have been carried out to date. These are the responses to changes in moderator water inlet temperature and fuel tube water inlet temperature. The purpose of these calculations was to estimate the iron temperature overshoot, if any, and the equations were even more simplified. The moderator was assumed to be connected to a separate flow circuit so that the mass flow of moderator water was constant. Only one delay group was used. Consideration of the boundary conditions imposed on the reactor by the propulsive machinery indicates that a flow factor of 1 is a reasonable value and this was used. The results are shown in figures 3 and 4. It is seen that the wall temperatures at no time exceed those calculated for the new equilibrium states of the reactor.

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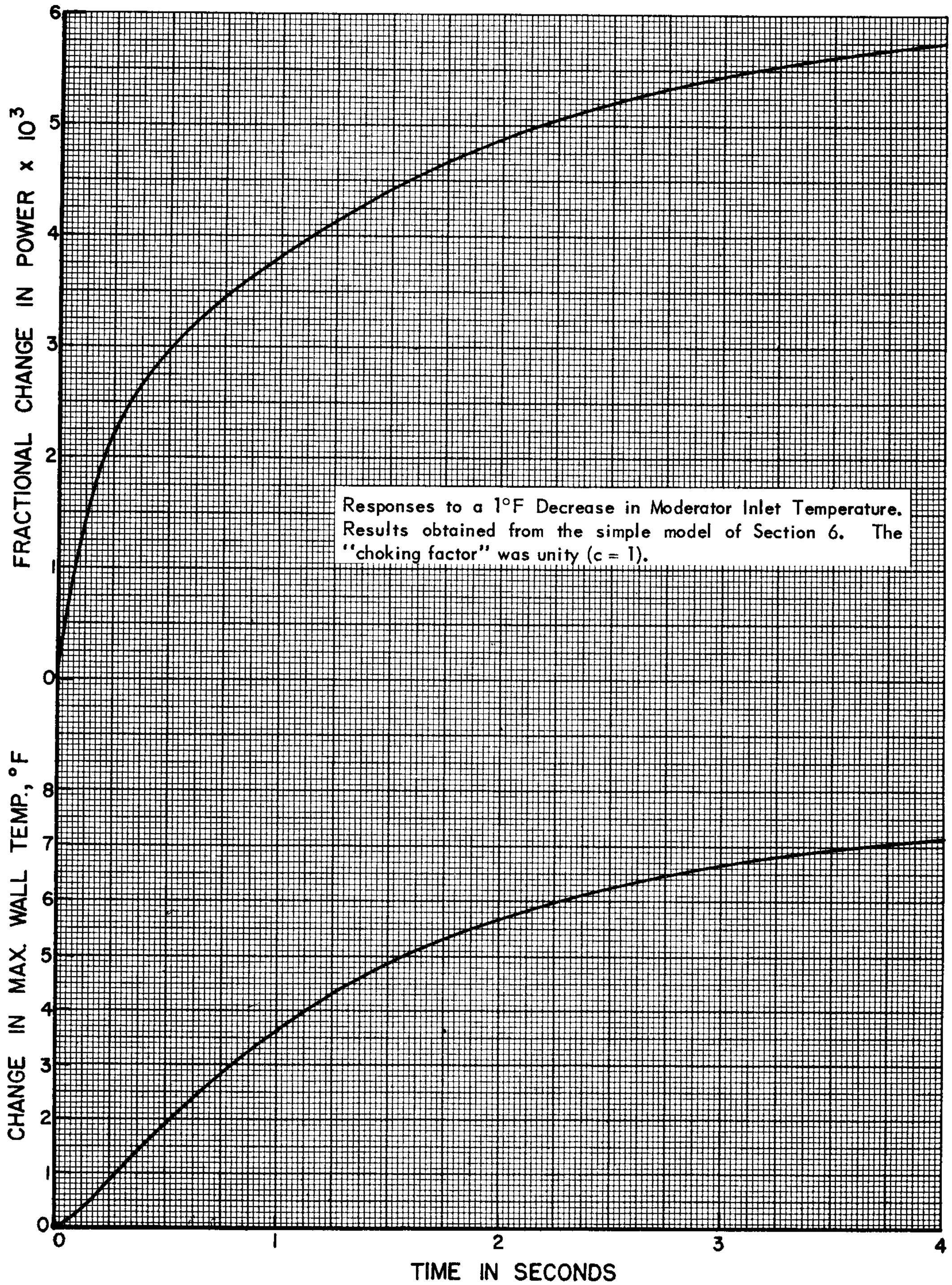

Responses to a 1°F Decrease in Moderator Inlet Temperature. Results obtained from the simple model of Section 6. The "choking factor" was unity (c = 1).

FIG. 3
12-TR-3

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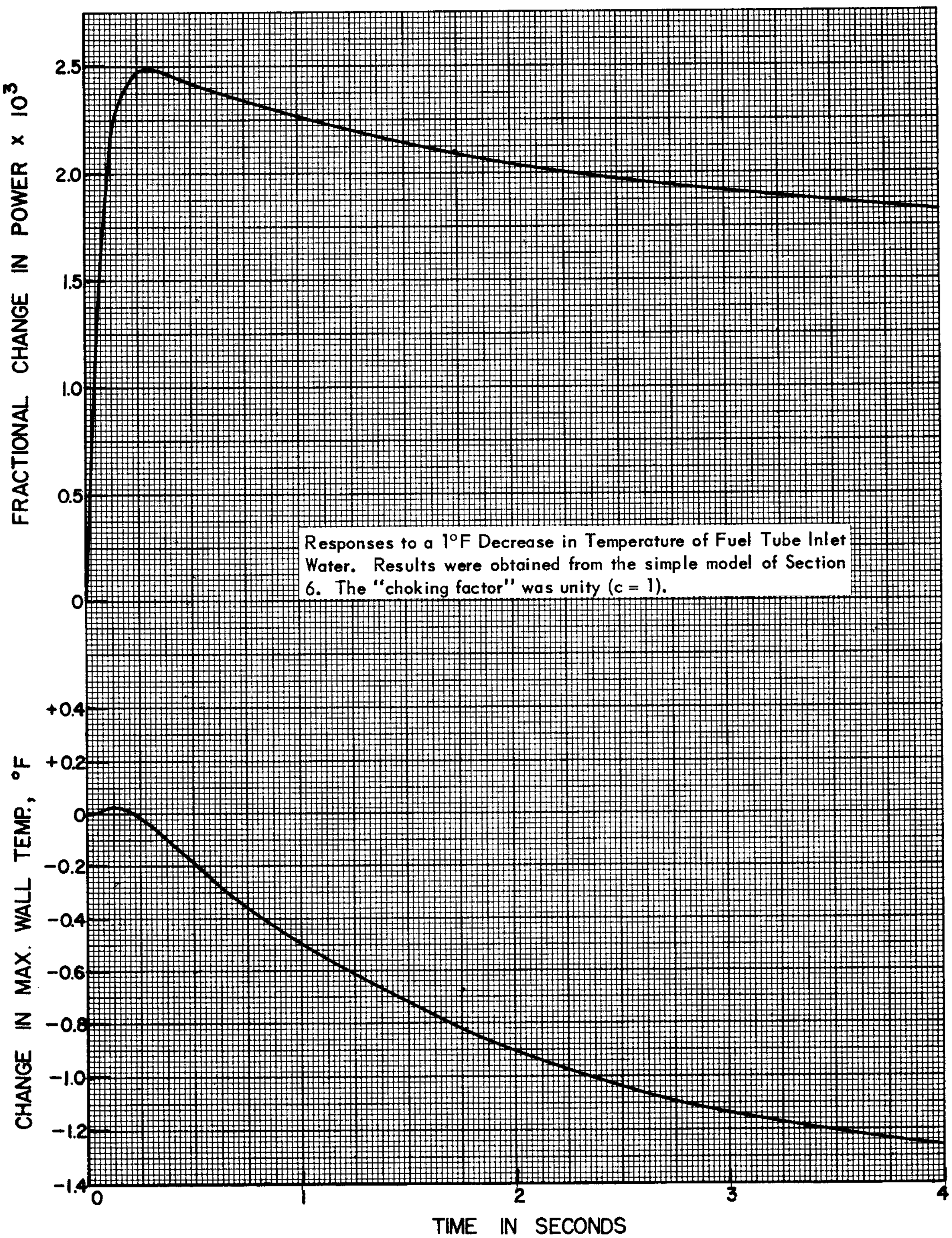

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FIG. 4
12-TR-4

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Section 6.3 MODERATOR BY-PASS VALVE

The detailed calculation was carried out under the assumption that the flow rate in the moderator and fuel elements was the same. The same assumption has been made in sections 6.1 and 6.2 and the only calculations made with the simpler model are subject to this hypothesis. For the purpose of shim control the reactor was designed so that the fraction of the total which passes through the moderating chamber is variable. By this means it is possible to vary the moderator density without varying the density distribution in the fuel tube appreciably. This flow scheme is shown schematically in figure 5.

For the purpose of completeness the equations of the simple model will be extended to include the case when f is not equal to unity. In this case the time required to change once completely the water in the moderating region will be $\frac{2.3}{f}$ seconds instead of 2.3 seconds, as before. The equilibrium value of ρ_m' will also be increased by the factor 1/f. This leads to a modification of equation 6.7.

$$\frac{2.3}{f}\frac{d}{dt}\frac{\rho_m'}{(\rho_m')_1} + \frac{\rho_m'}{(\rho_m')_1} = \frac{1}{f}\left(\frac{n'}{n^o} - \frac{g'}{g^o}\right) + \frac{{\rho_m'}^{in}}{(\rho_m')_1} \qquad (6.16)$$

Here, as before, $(\rho_m')_1$ is the equilibrium value of ρ_m'

[redacted]

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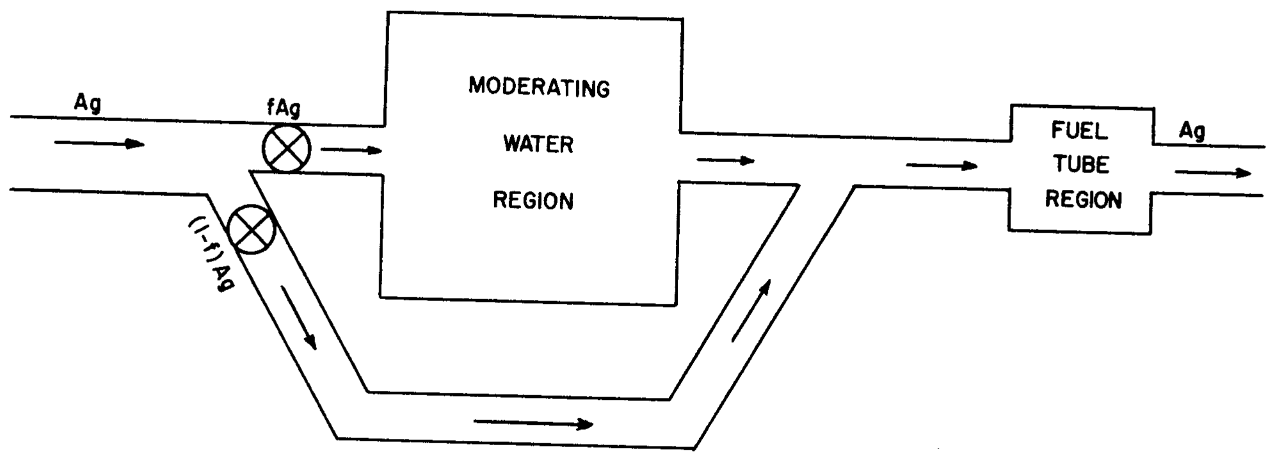

FIGURE 5

SCHEMATIC FLOW DIAGRAM

[redacted]

[redacted]

for a unit fractional power change ($n' = n^o$) with $f = 1$, other variables being held constant ($g' = \rho_m'^{in} = 0$).

[redacted]

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APPENDIX I

The matrices M and m which determine the inverse periods through the secular equation $|M - \lambda m| = 0$ are given in Appendix VI. The solution of this equation gives the following values for the roots: -3.14×10^{-3}; -4.5×10^{-2}; -6.24×10^{-1}; -1.547; -9.087; -52.8; -114.8; -223 sec^{-1}.

The homogeneous equation (5.3) determines, except for a normalizing factor, a set of u_j corresponding to each value of λ. Each such set gives a normal mode when inserted into the trial function form on page 41. These normal modes together with similarly obtained adjoint normal modes are used in solving time behavior problems by standard procedures.

Figures 2 and 6 graph the time response of the neutron density and maximum steel wall temperature obtained by such detailed calculation for a step change in reactivity and for a linear change in reactivity. Also plotted in Figure 2 are the neutron responses to the same step change in reactivity as computed according to the simple reactor models discussed in Section 6. Figures 4 and 3 present the response of the neutron density and maximum steel wall temperature following step changes in the water temperature at the inlet to the fuel tube and the inlet to the moderating chamber.

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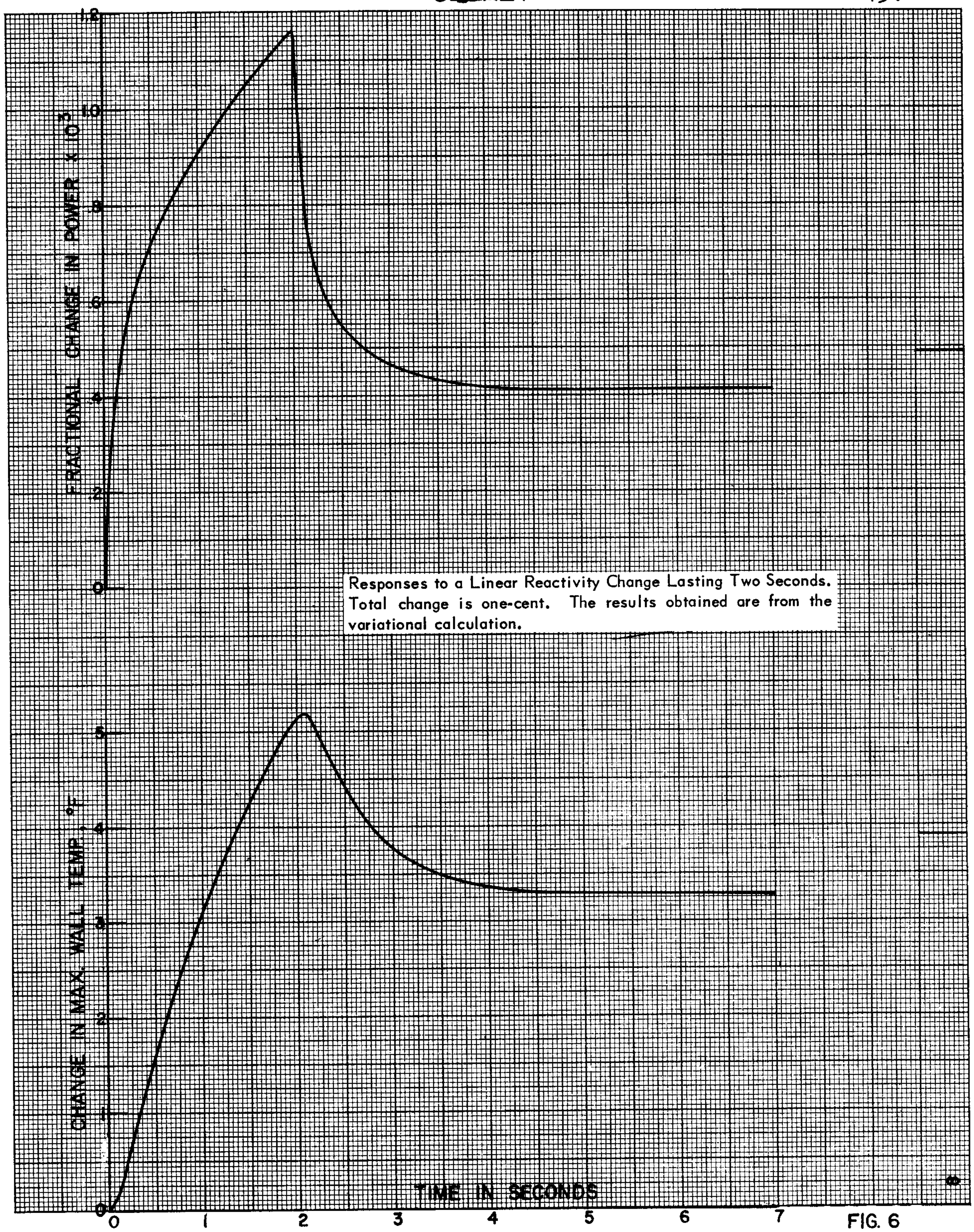


Responses to a Linear Reactivity Change Lasting Two Seconds. Total change is one-cent. The results obtained are from the variational calculation.

FIG. 6
12-TR-6

[redacted]

[redacted]

[redacted]

APPENDIX II

Steady State Neutron Data

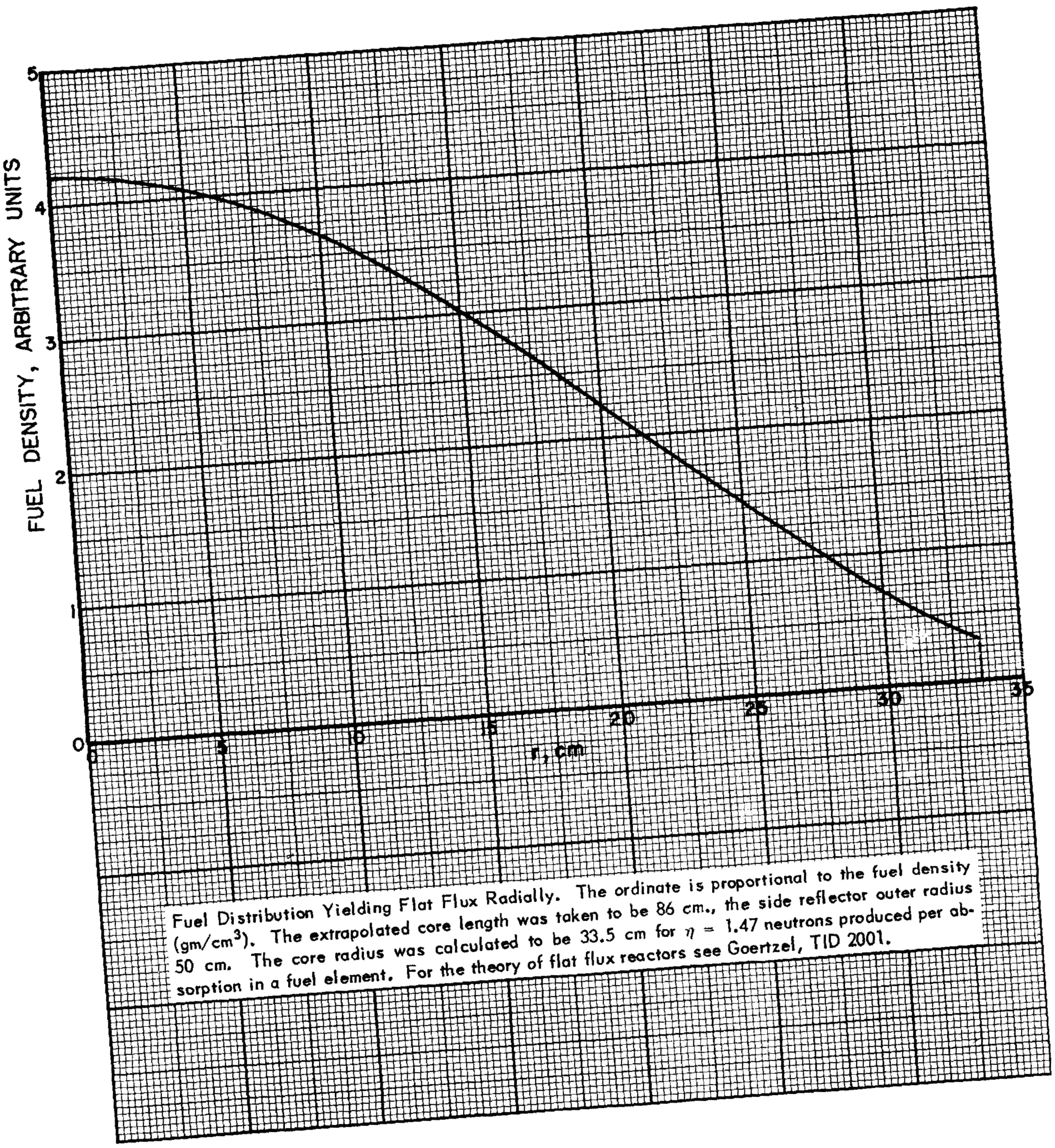
The reactor is idealized to consist of a cylindrical loaded core, radius 33.5 cm, actual height 75 cm and surrounded on its lateral surface by a cylindrical water reflector of outer radius 50 cm. In order to take into account the actual reflection present at both end faces of the core, an extrapolation length of 5.5 cm at each end is taken. The neutron density along the cylinder axis is taken to vary like the sine of the distance and to vanish at the extrapolated end points. When the fuel is distributed according to Figure 7, the resultant reactor has a thermal flux which is uniform throughout the core in the radial direction. The core contains 20 kg of U(25), 310 kg of stainless steel distributed in the form of several hundred identical fuel tubes. In computing the steady state neutron distribution (and fuel distribution) the water is taken to be at an effective density of .67 gm/cc, and an effective temperature of 295° C.

The important two group constants are tabulated below.

[redacted]

[redacted]

[redacted]



Fuel Distribution Yielding Flat Flux Radially. The ordinate is proportional to the fuel density (gm/cm^3). The extrapolated core length was taken to be 86 cm., the side reflector outer radius 50 cm. The core radius was calculated to be 33.5 cm for η = 1.47 neutrons produced per absorption in a fuel element. For the theory of flat flux reactors see Goertzel, TID 2001.

FIG. 7
12-TR-7

[redacted]

[redacted]

TABLE II.I

Nuclear Values*

	Value at Room Temperature	Dependence on Neutron Energy	Value at ρ=0.667gm/cc T=292° C
Absorption Cross Sections			
σ_{Fe}	0.0258 cm^2/gm	1/v	0.0186 cm^2/gm
σ_{25}	1.639 cm^2/gm	1/v	1.180 cm^2/gm
σ_{water}	0.0227 cm^2/gm	1/v	0.0163 cm^2/gm
Diffusion Lengths			
a_s	2.71 cm	v	5.66 cm
a_f	5.0 cm	const.	7.5 cm
Neutrons per absorption in fuel			
η_o	2.10	const.	
η	1.47	const.	
Delayed neutron Parameters			
λ_1	0.0464 sec^{-1}	const.	
λ_2	0.602 sec^{-1}	const.	
β_1	0.404x10^{-2}	const.	
β_2	0.351x10^{-2}	const.	

*These are the values employed in the calculation and are in some cases not the best data available at this time.

[redacted]

APPENDIX III

Steady State Flow Data

The relevant steady state flow data for the design point in question is given below.

Total power = 400,000 kw.
Total water flow rate = 429.2 #/sec
(all through moderating chamber and then into fuel tubes)

TABLE III.1

Steady State Flow Data

	Inlet	Moderating Chamber	Exit From Fuel Tubes
Pressure	P_L+1.73 kg/cm^2 (24.6 psi)	P_L+1.73 kg/cm^2 (24.6 psi)	P_L = 352 kg/cm^2 (5000 psi)
Temp.	249° C (480°F)	(292° C) (558° F)	527° C (980° F)
Density	0.768 gm/cc (47.9 #/ft^3)	0.768 gm/cc (47.9 #/ft^3)	0.128 gm/cc (7.96 #/ft^3)

The variations of pressure and temperature of the fuel tube water and the variations of the steel temperature along the fuel tube direction are presented in Figures 8 and 9.

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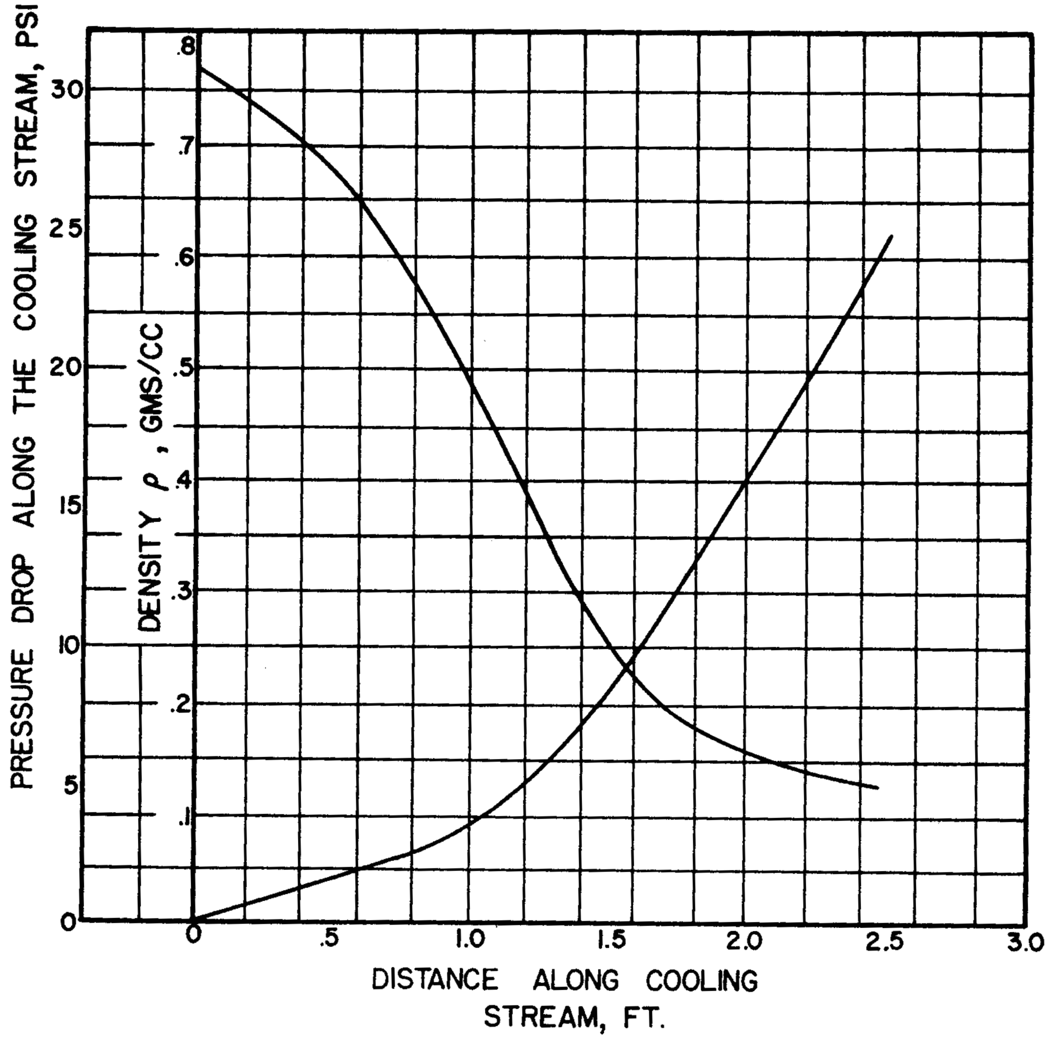
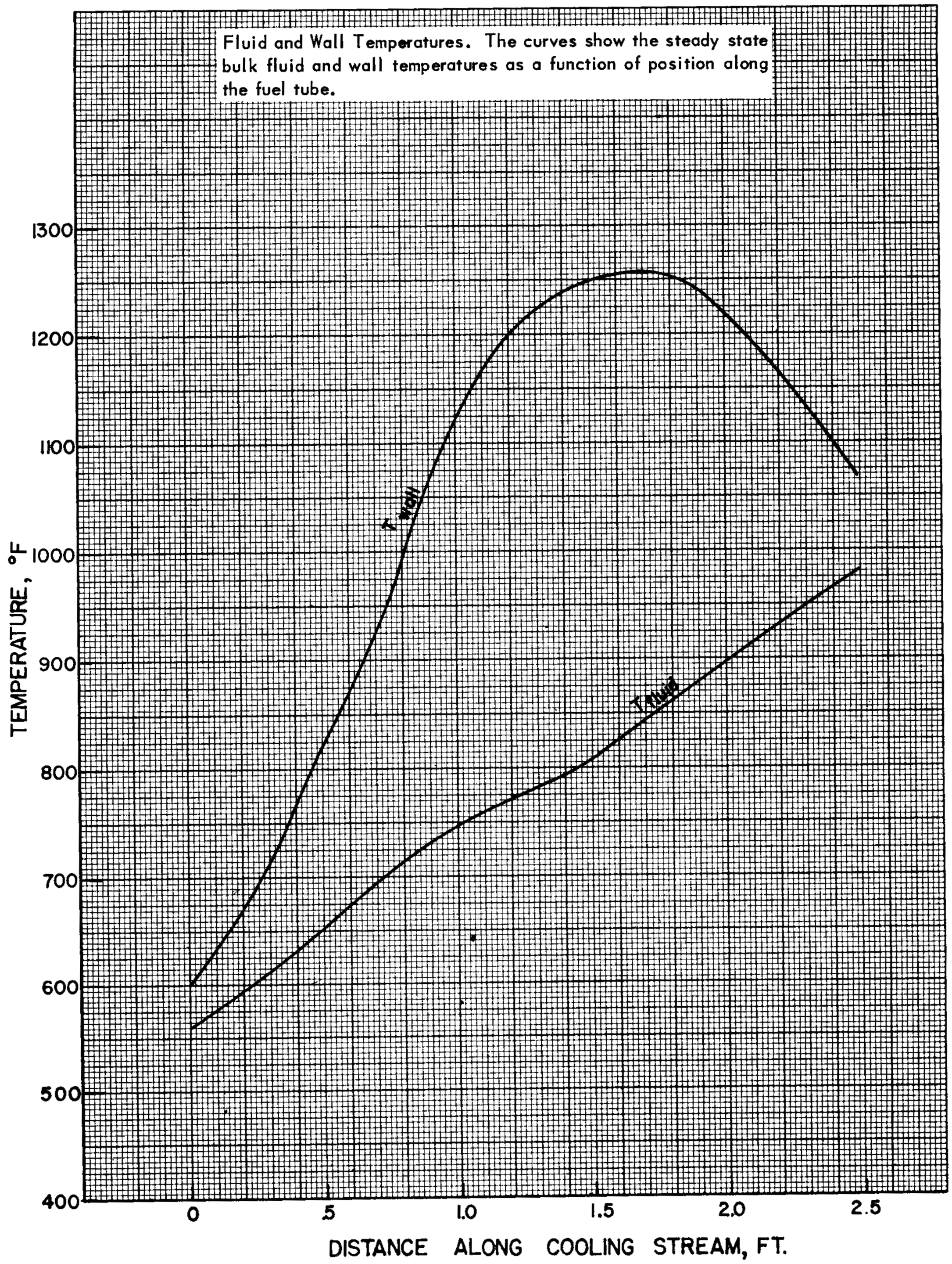


Longitudinal Pressure and Density Variations. The curve shows the steady state variation of water pressure and density along the length of a fuel tube.

FIG. 8
12-TR-8

[redacted]

[redacted]



Fluid and Wall Temperatures. The curves show the steady state bulk fluid and wall temperatures as a function of position along the fuel tube.

FIG. 9
12-TR-9

[redacted]

[redacted]

[redacted]

APPENDIX IV

The calculation of the γ ray heating is summarized in the following table.

TABLE IV.1

Gamma Ray Heating

Source	Energy of Individual γ-ray (Mev)	Total γ-ray energy per single fission	Fraction absorbed in core iron	Fraction absorbed in core water	Fraction absorbed outside of core
Prompt	5	5	1/3	1/3	1/3
Fission Product	4	4	1/3	1/3	1/3
Capture: U(25)	7	1.75	.28	.28	.44
Fe	7	3.75	.28	.28	.44
H_2O	2.2	0.62	.30	.30	.40

In addition to being heated by the gamma rays the water receives the kinetic energy of the fission neutrons. This is estimated to be 5 Mev/fission. The gamma ray plus neutron heating of the water amounts to

$$5+\left[5\left(\frac{1}{3}+\frac{1}{3}\right)+4\left(\frac{1}{3}+\frac{1}{3}\right)+1.75(.28+.44)+3.75(.28+.44)+0.62(.30+.40)\right]=15.4$$

Mev/fission or 7.6% of the pile power. This is .076(400,000) = 30,400 kw. The conduction heating from the outside of the fuel elements brings this up to a total of 40,000 kw. 80% of this

[redacted]

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total is estimated to appear promptly, the remainder being delayed several or more seconds.

The γ- heat into the core iron is 4.7 Mev/fission. This amounts to 9400 kw. Thus

$$q_m{}^{o} = 0.1 \text{ (total power)}$$

and

$$q_m{}^{(p)} = q_o \int n d\tau = 0.08 \frac{n}{n^o} \text{ (total power)}$$

considering only the prompt variation of q_m with n. Also

$$q_I{}^{o} = 0.90 \text{ (total power)}$$

of which the fractions $\frac{360000 - 9400}{360000} = 0.975$ is composed of direct fission heat, and the fraction 0.025 of γ-ray heat, most of the latter being prompt. Thus

$$q_1\, n^o = 0.9(0.975)\text{(total power)}\; \frac{\sin \pi \dfrac{z+5.5\text{cm}}{4+11 \text{ cm}}}{\int_0^L \sin\pi \dfrac{z+5.5\text{cm}}{L+11 \text{ cm}}\, dz}$$

$$q_2 \int n d\tau = \frac{0.9(0.025)}{L} \text{ (total power)}$$

where L = 75 cm, the length of the fuel tubes.

[redacted]

[redacted]

APPENDIX V

The Trial Functions

The functions in terms of which the trial functions are determined are given here.

Neutrons $n^{o} = \text{const} \sin \frac{\pi(z+5.5\,\text{cm})}{86\ \text{cm}} f(r)$

where $f(r)$ is flat over the core.

n^{o} is a solution of

$$(K_f \eta y n^{o})^{o} - (y + w\alpha_s)^{o}\, n^{o} = 0$$

$$N^{o} = \text{const} \sin \frac{\pi(z+5.5\ \text{cm})}{86\ \text{cm}} F(r)$$

and is a solution of

$$\alpha_s{}^{o} N^{o} = y n^{o}$$

Delays:

$$c_i{}^{o} = \frac{\beta_i \eta^{o} y n^{o}}{\lambda_i}$$

where in the approximation of two delay groups, $\beta_1, \beta_2, \lambda_1, \lambda_2$ are given in Table II.1.

Xenon

$$x^{o} = 4.11 \times 10^{10} \frac{\text{sec}}{\text{cm}^3}\, y^{o}$$

corresponding to

[redacted]

[redacted]

$$\overline{v\sigma_x} = 8.33 \times 10^{-13} \frac{cm^3}{sec} \; ; \; n_{max}{}^{o} \, \overline{v\sigma_x} = 4.76 \times 10^{-3} \ sec^{-1}$$

fission yield = 5.8×10^{-2} xenon atoms/fission

$$b = 1.71 \times 10^{-3} \, y^{o}.$$

This expression for b results from the assumption that 5% of the xenon is produced very quickly after fission, the remaining 95% passing through the comparatively long lived iodine intermediary. In the matrix, it is required to use the fact that

$$\frac{\partial y}{\partial x} = \overline{v\sigma_x}$$

Flow

Mode 1 and 2 solutions are presented in Figures 10, 11, 12, 13.

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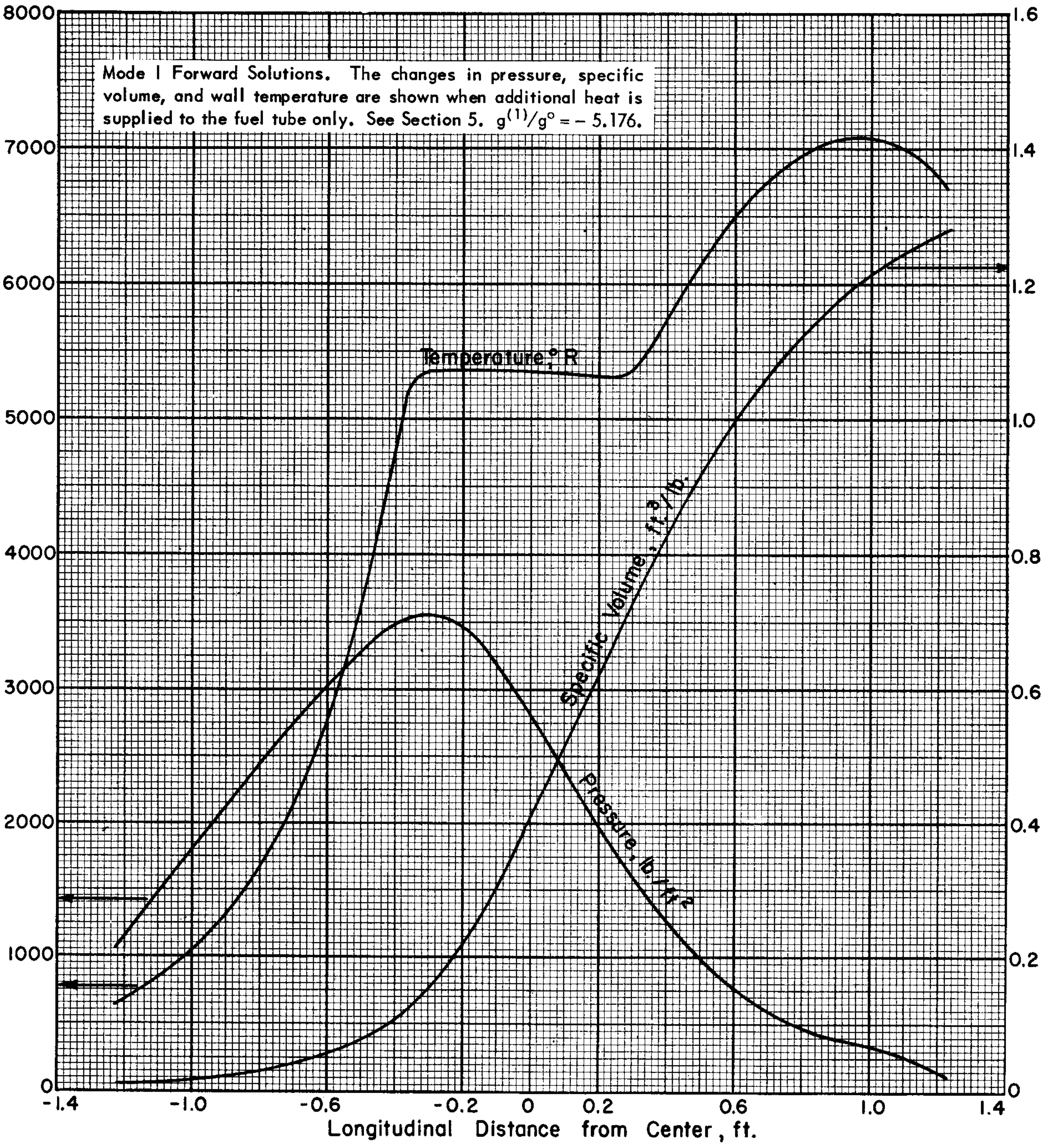
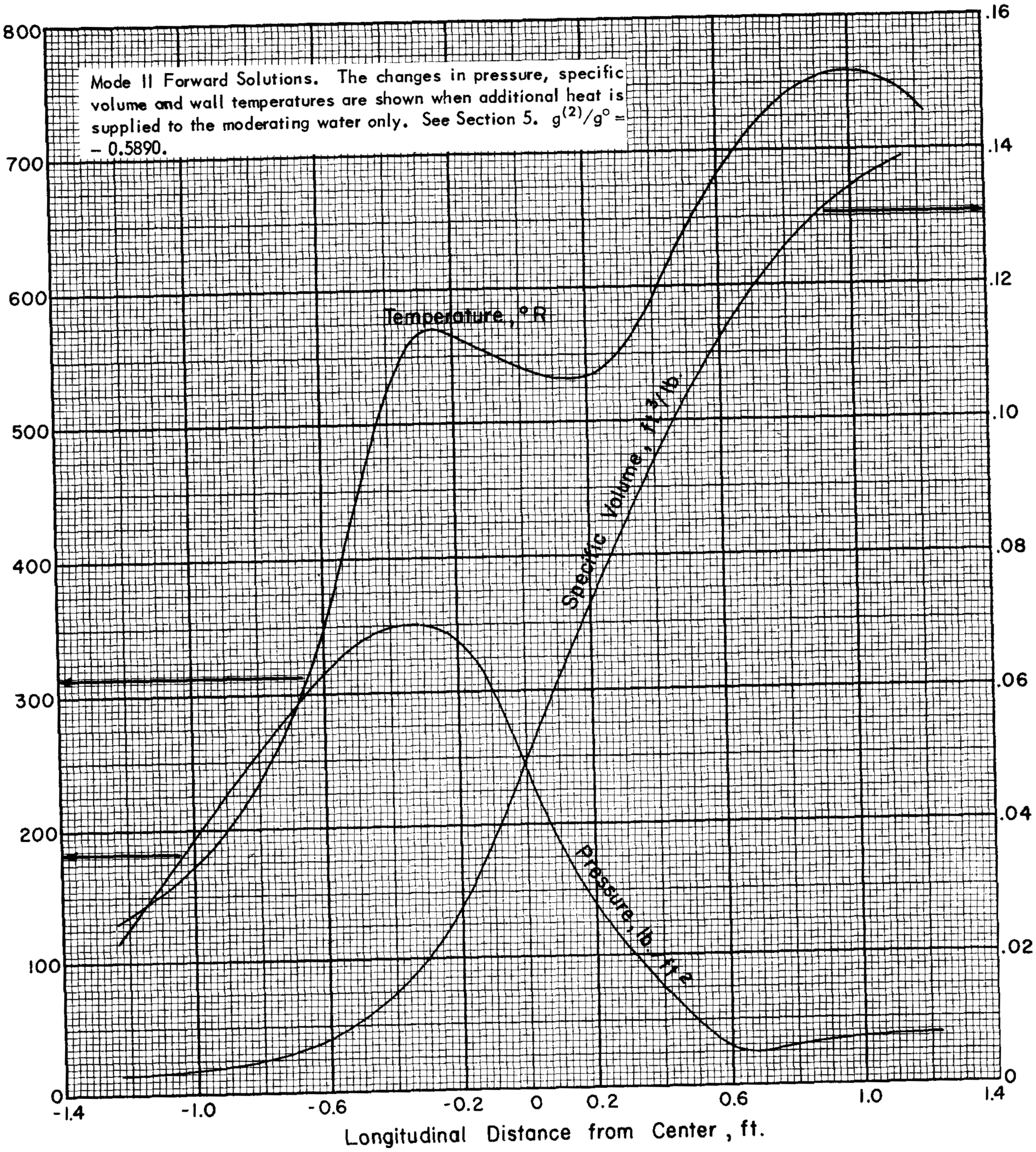


[redacted]

FIG. 10
12-TR-10

[redacted]



[redacted]

FIG. 11

12-TR-11

[redacted]

Mode I Adjoint Solutions. Changes in the adjoint functions are shown when additional heat is supplied to the fuel tube only. See Section 5. $G^{(1)} = .00670$ sec. ft./#.

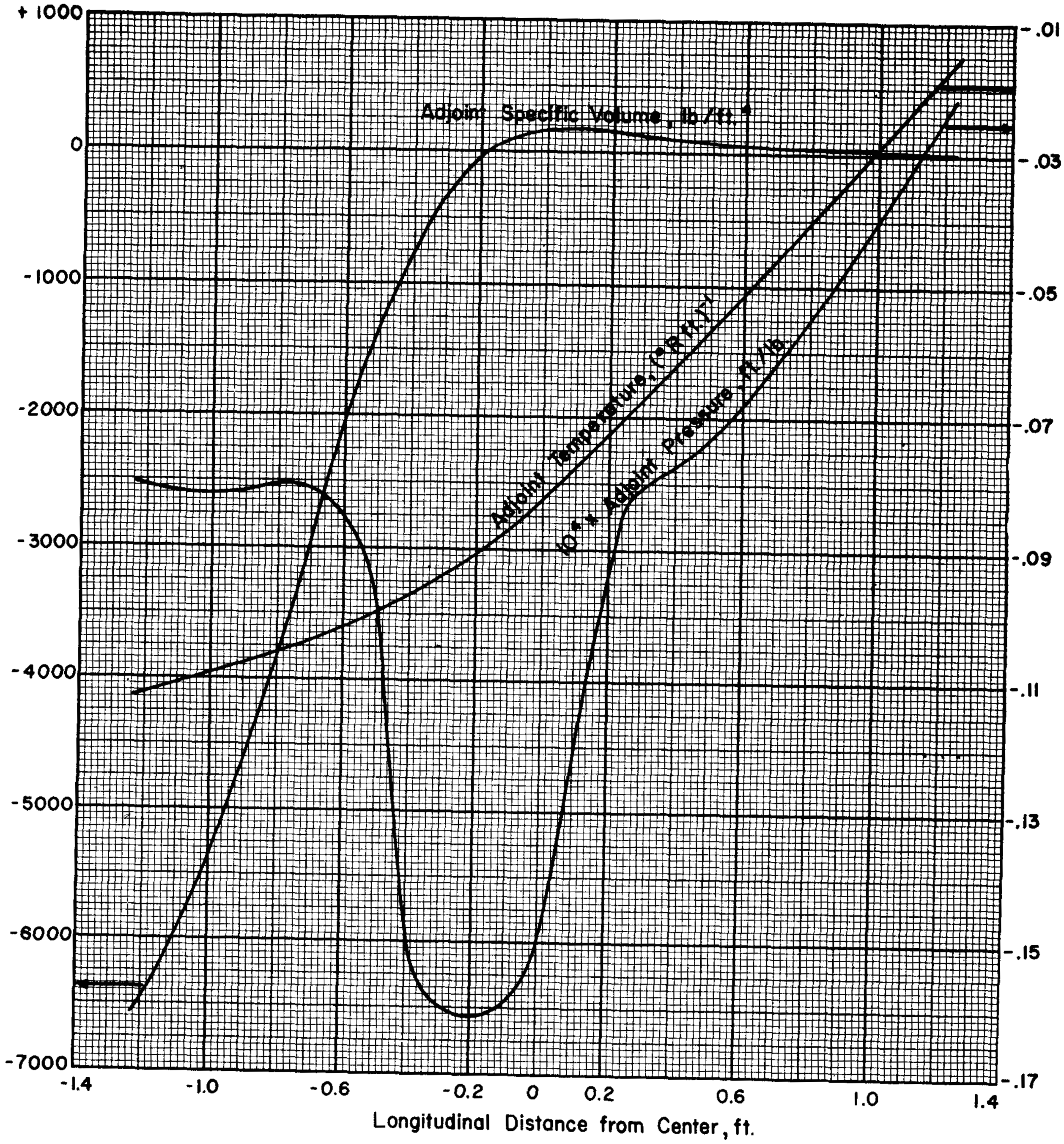


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FIG. 12

12-TR-12

[redacted]

[redacted]

Mode II Adjoint Solutions. Changes in the adjoint functions are shown when additional heat is supplied to the moderator water only. See Section 5. $G^{(2)} = .0163$ sec. ft./#.

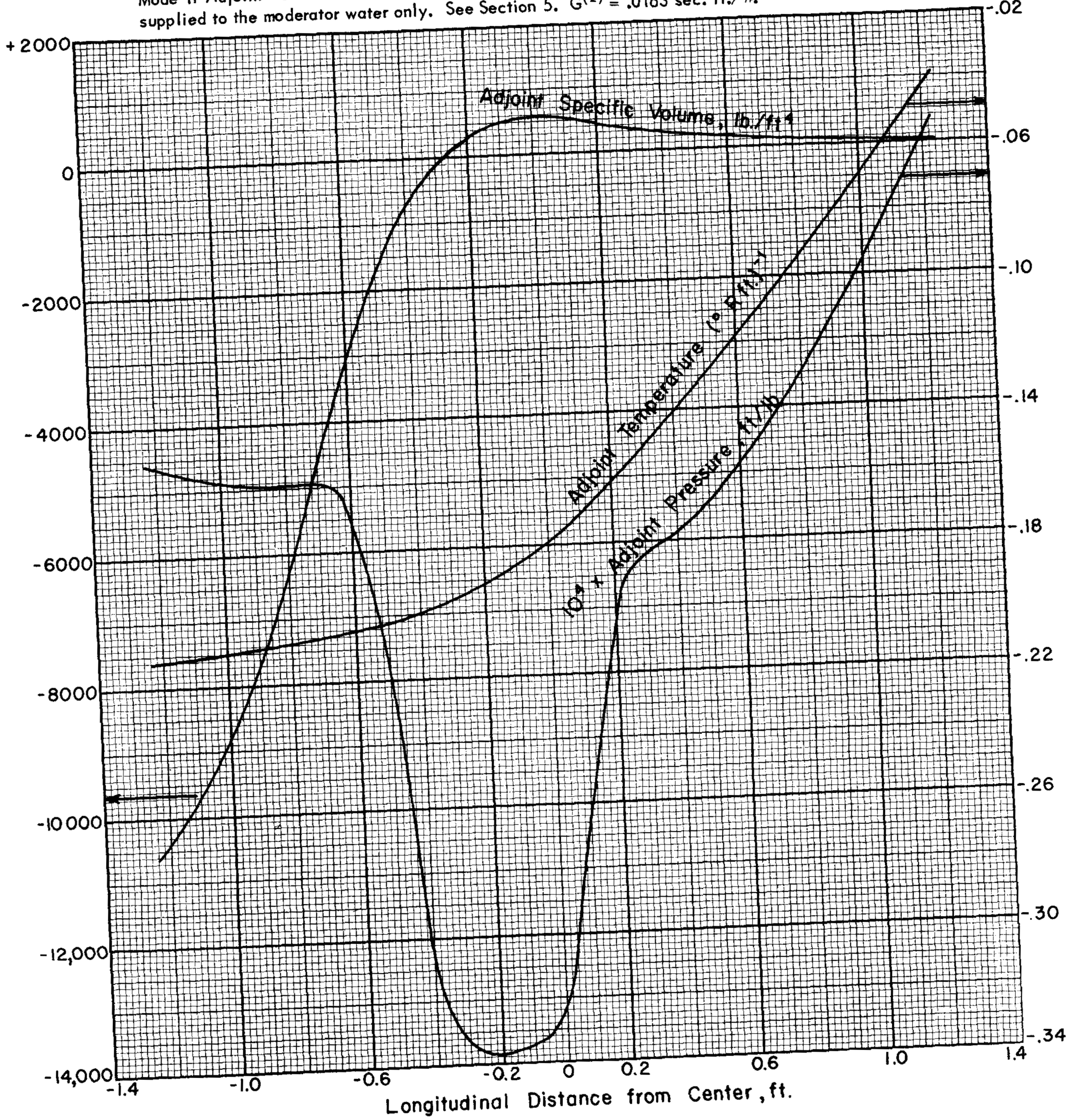

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FIG. 13

12-TR-13

[redacted]

[redacted]

APPENDIX VI

Matrix Elements

The matrix m is an almost diagonal matrix. It would be diagonal except for the split of the thermodynamics into two modes. The non-zero elements of m are

$$m_{11} = \int N^{o} n^{o} d\tau$$

$$m_{22} = \int c_1{}^{o} K_f{}^{o} N^{o} d\tau$$

$$m_{33} = \int c_2{}^{o} K_f{}^{o} N^{o} d\tau$$

$$m_{44} = \int x^{o} N^{o} d\tau$$

$$m_{55} = \rho_m{}^{(1)} R_m{}^{(1)} + \int \rho^{(1)} R^{(1)} dz + \int p^{(1)} P^{(1)} dz + \int g^{(1)} G^{(1)} dz$$

$$m_{56} = \rho_m{}^{(2)} R_m{}^{(1)} + \int \rho^{(2)} R^{(1)} dz + \int p^{(2)} P^{(1)} dz + \int g^{(2)} G^{(1)} dz$$

$$m_{65} = \rho_m{}^{(1)} R_m{}^{(2)} + \int \rho^{(1)} R^{(2)} dz + \int p^{(1)} P^{(2)} dz + \int g^{(1)} G^{(2)} dz$$

$$m_{66} = \rho_m{}^{(2)} R_m{}^{(2)} + \int \rho^{(2)} R^{(2)} dz + \int p^{(2)} P^{(2)} dz + \int g^{(2)} G^{(2)} dz$$

$$m_{77} = \int \theta^{(1)} \Theta^{(1)} dz$$

$$m_{78} = \int \theta^{(2)} \Theta^{(1)} dz$$

$$m_{87} = \int \theta^{(1)} \Theta^{(2)} dz$$

$$m_{88} = \int \theta^{(2)} \Theta^{(2)} dz$$

[redacted]

[redacted]

[redacted]

Some of the elements of the matrix M are given below; as examples of the procedure outlined in Section V.

$$M_{11} = \int N^\circ \left[K_f(1-\beta)\eta y - y - w\mathcal{L}_s \right]^\circ n^\circ d\tau$$

$$M_{15} = \left[\int N^\circ \left\{ \left(\frac{\partial K_f}{\partial \alpha_f}(1-\beta)\eta y n \frac{\partial \alpha_f}{\partial \bar{\rho}} - \frac{\partial w}{\partial \bar{\rho}} \mathcal{L}_s - w \frac{\partial \mathcal{L}_s}{\partial \alpha_s} \frac{\partial \alpha_s}{\partial \bar{\rho}} \right) \frac{\partial \bar{\rho}}{\partial \rho_m} - w \frac{\partial \mathcal{L}_s}{\partial \alpha_s} \frac{\partial \alpha_s}{\partial \bar{T}} \frac{\partial \bar{T}}{\partial \rho_m} \right\}^\circ d\tau \right] \rho_m^{(1)}$$

$$+ \int N^\circ \left\{ \frac{\partial K_f}{\partial \alpha_f} (1-\beta)\eta y n \frac{\partial \alpha_f}{\partial \bar{\rho}} - \frac{\partial w}{\partial \bar{\rho}} \mathcal{L}_s - w \frac{\partial \mathcal{L}_s}{\partial \alpha_s} \frac{\partial \alpha_s}{\partial \bar{\rho}} \right\}^\circ d\tau \int \left(\frac{\partial \bar{\rho}}{\partial \rho} \right)^\circ \rho^{(1)} dz$$

$$- \int N^\circ \left(w \frac{\partial \mathcal{L}_s}{\partial \alpha_s} \frac{\partial \alpha_s}{\partial \bar{T}} \right)^\circ d\tau \int \left(\frac{\partial \bar{T}}{\partial \rho} \right)^\circ \rho^{(1)} dz$$

$$M_{17} = M_{18} = 0$$

$$M_{71} = \int \Theta^{(1)} \frac{q_1}{c} n^\circ dz + \int \frac{\Theta^{(1)}}{c} dz \int q_2 n^\circ d\tau$$

$$M_{75} = \int \frac{\Theta^{(1)}}{c} \left(\frac{\partial J}{\partial T} \frac{\partial T}{\partial \rho} \right)^\circ \rho^{(1)} d\tau - \int \frac{\Theta^{(1)}}{c} \left(\frac{\partial J}{\partial g} \right)^\circ g^{(1)} d\tau$$

$$M_{77} = - \int \frac{\Theta^{(1)}}{c} \left(\frac{\partial J}{\partial \theta} \right)^\circ \theta^{(1)} d\tau$$

$$M_{78} = - \int \frac{\Theta^{(1)}}{c} \left(\frac{\partial J}{\partial \theta} \right)^\circ \theta^{(2)} d\tau$$

$$M_{72} = M_{73} = M_{74} = 0$$

[redacted]

[redacted]

When all matrix elements are evaluated, the result is:

m

1	0	0	0	0	0	0	0
0	1858	0	0	0	0	0	0
0	0	124.7	0	0	0	0	0
0	0	0	1.64×10^5	0	0	0	0
0	0	0	0	-1814	-483.8	0	0
0	0	0	0	-7808	-2102	0	0
0	0	0	0	0	0	-732.9	-81.77
0	0	0	0	0	0	-1590	-176.3

M

-161.5	86.4	75.1	612	-3148	-702.8	0	0
86.4	-86.4	0	0	0	0	0	0
75.1	0	-75.1	0	0	0	0	0
612	0	0	-612	0	0	0	0
-129.6	0	0	0	10967	1442	-10967	-1312
-574.1	0	0	0	23440	3335	-23440	-2761
-1015.3	0	0	0	-9952	-1312	10,967	1312
-2146	0	0	0	-21294	-2761	23,440	2761

[redacted]