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OAK RIDGE SCHOOL OF REACTOR TECHNOLOGY

SIMPLIFIED TWO-GROUP CALCULATION FOR REFLECTED HOMOGENEOUS THERMAL REACTORS

From Lectures by T. A. Welton, March 1952, ORSORT

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1. Introduction

The method here presented enables rapid solution of a certain restricted class of reflected homogeneous reactor problems, within reasonable limits of accuracy. It is particularly useful for preliminary calculations of reflected homogeneous reactors, and for estimation of safe dimensions of tanks, pipes, etc. designed to hold or carry solutions or slurries of fissile material. The simplicity obtains only at the expense of some generality, so that the following limitations hold:

(1) Core moderator and reflector material are identical.

(2) Atomic ratio of fuel-to-moderator in core is small enough so that both core and reflector have essentially the same transport properties, i.e., diffusion coefficient and scattering cross section.

(3) All fissions are assumed to be induced by thermal neutrons only.

(4) In the case of a reflected reactor, the reflector thickness is sufficiently great to be treated as infinite.

We shall assume two groups. Where necessary, we shall let subscript 1 refer to fast group quantities and subscript 2 to the thermal group properties.

2. Constants Needed

Upon selecting a moderating material and a fuel material we shall need only the following constants:

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(1) τ = Fermi "Age" of thermal neutrons from fission source in the chosen moderator-reflector medium.

(2) L = Thermal diffusion length in 100% moderator (hence in reflector).

(3) $D = \frac{1}{3\Sigma_{tr}}$ = Diffusion coefficient for thermal neutron diffusion, assumed to be the same in both core and reflector.

(4) η = Average number of fission neutrons emitted per thermal neutron absorption (including capture) by a fuel nucleus.

(5) $\alpha = \frac{A_m \sigma_u}{A_u \sigma_m} = \frac{\text{At.Wt. moderator times tot. abs. cross sec. fuel}}{\text{At.Wt. fuel times abs. cross sec. moderator}}$

In addition, of course, one must choose the geometry and know either the dimensions of the system or the concentration C of fissile (fuel) material. In the following treatment we shall assume that the dimensions are known and shall solve for the concentration C yielding criticality.

3. Derived and Defined Quantities

Let us define the following quantities:

$$C = \text{fuel concentration} = \frac{\text{kilograms of fuel atoms}}{\text{liter of solution}} \text{ or } \frac{\text{grams fuel}}{\text{cm}^3 \text{ mixture}} \quad (1)$$

$\Sigma_R = \frac{D}{L^2}$ = macroscopic absorption cross-section for thermal neutrons in the reflector (i.e., in 100% moderator) (2)

Σ_C = macroscopic absorption cross-section for thermal neutrons in the core (3)

Σ_U = macroscopic absorption (capture plus fission) cross section for thermal neutrons by fuel (4)

Evidently

$$\Sigma_C = \Sigma_U + \Sigma_R = \Sigma_R \left[1 + \frac{\Sigma_U}{\Sigma_R} \right] = \Sigma_R \left[1 + \frac{C \sigma_u A_m}{\sigma_m A_u} \right] = \Sigma_R (1 + \alpha C) \quad (5)$$

$$\Sigma_U = \alpha C \Sigma_R \quad (6)$$

$$f = \frac{\Sigma_U}{\Sigma_U + \Sigma_R} = \frac{\alpha C}{1 + \alpha C} = \text{thermal utilization factor} \quad (7)$$

$$L_C^2 = \frac{D}{\Sigma_C} = \frac{L^2 \Sigma_R}{(1 + \alpha C)\Sigma_R} = \frac{L^2}{1 + \alpha C} = \text{square of thermal diffusion length in the core} \quad (8)$$

We now proceed with our two-group analysis, assuming the validity of a one-group description of the moderation process theory and diffusion theory for the thermal neutron. For the sake of definiteness we shall assume spherical geometry; however, the treatment may be applied to other geometries without introducing new concepts.

Consider first the fast group. Neutrons are "absorbed" from the fast group by slowing down plus moderator capture, in a manner spatially describable in terms of the experimental quantity τ, the Fermi age. Since we are postulating all fissions to occur by absorption from the thermal group only, the material properties for scattering and absorption of fast group neutrons are the same for both core and reflector, meaning that τ will be the same in the two regions. From τ we can obtain a fictitious diffusion coefficient and macroscopic "absorption" cross section for fast group neutrons by <u>defining</u>

$$\frac{D_1}{\Sigma_1} = \tau . \tag{9}$$

The problem under consideration can be viewed as a diffusion problem with sources. That is, a distributed source of fast neutrons exists in the core by virtue of fissions produced by thermals. Likewise, a distributed source of thermal neutrons exists in the core because of slowing down of fast neutrons. The procedure to be used makes use of certain "partial multiplication constants" k_1 and k_2, defined as follows:

k_1 = the number of fast neutrons which must be introduced in the core to give one fast neutron absorbed in the core.

k_2 = the number of thermal neutrons which must be introduced in the core to give one thermal neutron absorbed in the core.

Note that no unique answer for k_1 and k_2 exists from above definition, since shape of source is of some importance. In what follows, approximate values for k_1 and k_2 are obtained, assuming that each source has a "normal mode" shape.

To obtain k_1, write

$$D_1\nabla^2\phi_1(r) - \Sigma_1\phi_1 + k_1\Sigma_1\phi_1 = 0, \quad 0 \leq r \leq R \text{ (core)}$$
$$D_1\nabla^2\phi_1(r) - \Sigma_1\phi_1 = 0 \quad , \quad R \leq r < \infty \text{ (reflector)}$$

with $\phi_1(0)$ finite and $\phi_1(\infty) = 0$.

Defining $B_1^2 = \dfrac{(k_1 - 1)\Sigma_1}{D_1} = \dfrac{k_1 - 1}{\tau}$ (12)

our equations are

$$\nabla^2\phi_1 + B_1^2\phi_1 = 0 \quad \text{(core)}$$
$$\nabla^2\phi_1 - \frac{1}{\sqrt{\tau}}\phi_1 = 0 \quad \text{(reflector)}$$

whence

$$\phi_1(r) = \begin{cases} \dfrac{a_1 \sin B_1 r}{r} , & 0 \leq r \leq R \\ \dfrac{b_1 e^{-r/\sqrt{\tau}}}{r} , & R \leq r < \infty \end{cases} \tag{13}$$

At the core-reflector interface (r=R) we have the usual continuity conditions for neutron flux and current; applied to the fast group this is

$$D_1\left[\frac{d}{dr}\ln\phi_1(r)\right]_{r=R-0} = D_1\left[\frac{d}{dr}\ln\phi_1(r)\right]_{r=R+0}$$

whence

$$B_1R \operatorname{ctn} B_1R = -\frac{R}{\sqrt{\tau}} \quad \text{determines } B_1 \tag{14}$$

while by (12)

$$k_1 = 1 + B_1^2\tau \text{ determines } k_1 \tag{15}$$

Similarly, to obtain k_2, write

$$D\nabla^2\phi_2(r) - \Sigma_c\phi_2 + k_2\Sigma_c\phi_2 = 0, \quad 0 \leq r \leq R \quad \text{(core)} \tag{16}$$

and for the reflector

$$D\nabla^2\phi_2(r) - \Sigma_R\phi_2 = 0, \quad R \leq r < \infty \quad \text{(reflector)} \tag{17}$$

Similarly

$$B_2R \text{ ctn } B_2R = -\frac{R}{L}, \qquad \text{determining } B_2, \tag{18}$$

and

$$k_2 = 1 + L_C^2 B_2^2, \quad \text{determining } k_2 \tag{19}$$

where we recall $L_C^2 = \dfrac{L^2}{1 + \alpha C}$ as given by (8).

We must now turn our attention to the matter of balancing the gain and loss of neutrons in each group for the core at criticality. To describe this we shall find it convenient to introduce the following quantities: first, the group absorption rates, (20)

A_1 = rate at which group 1 neutrons are being absorbed in the core,
A_2 = rate at which group 2 neutrons are being absorbed in the core, (20)

and the group source functions, (21), for the core (zero elsewhere):

$S(1\rightarrow 1)$ = number of fast neutrons produced in the core per fast neutron absorbed in the core
$S(2\rightarrow 1)$ = number of fast neutrons produced in the core per slow neutron absorbed in the core
$S(1\rightarrow 2)$ = number of slow neutrons produced in the core per fast neutron absorbed in the core
$S(2\rightarrow 2)$ = number of slow neutrons produced in the core per slow neutron absorbed in the core. (21)

According to our original assumptions we have

$$\begin{aligned} S(1\rightarrow 1) &= 0 \\ S(2\rightarrow 1) &= \eta f \\ S(1\rightarrow 2) &= 1 \quad \text{(approximately; actually} = p\epsilon) \\ S(2\rightarrow 2) &= 0 \end{aligned} \tag{22}$$

From the definitions (10) of k_1 and k_2 we have

$1/k_1$ = number of neutrons absorbed from the fast group per neutron entering the fast group, at criticality

$1/k_2$ = number of neutrons absorbed from the thermal group per neutron entering the thermal group, at criticality.

Hence the absorption rate in the core for fast group neutrons must satisfy

$$A_1 = \frac{1}{k_1}\left[S(1\rightarrow 1)A_1 + S(2\rightarrow 1)A_2 \right] \tag{23}$$

and that for thermal group neutrons must satisfy

$$A_2 = \frac{1}{k_2}\left[S(1\rightarrow 2)A_1 + S(2\rightarrow 2)A_2\right] + \text{(contribution to } A_2 \text{ due to reflector)} \tag{24}$$

where the second term on the right-hand side of (24) takes account of the absorption rate in the core of thermal neutrons which are born in the reflector from fast neutrons moderated there, and then wander back to the core and are absorbed. We shall evaluate this term in paragraph 5.

4. <u>The Bare Reactor</u>

If the reactor is bare the second term on the right-hand side of (24) is of course zero. Placing the values (22) of the source functions into (23) and (24) and collecting terms we then would have the following pair of simultaneous equations in A_1 and A_2:

$$k_1A_1 - \eta f A_2 = 0$$

$$A_1 - k_2A_2 = 0$$

requiring, for A_1 and A_2 not identically zero

$$\begin{vmatrix} k_1 & -\eta f \\ 1 & -k_2 \end{vmatrix} = 0$$

$$\text{or,} \quad k_1k_2 = \eta f = (1 + B_1^2\tau)\,(1 + B_2^2\,L_c^2).$$

Since in this case (bare core) the fluxes must vanish at the extrapolated core boundary, (14) and (18) do not apply, but rather $B_1 = B_2 = \frac{\pi}{R+d}$, and we have the familiar result for criticality of a bare reactor under the approximations assumed at the outset, viz:

$$\frac{\eta f}{(1 + B^2\tau)(1 + L_c^2\,B^2)} = 1$$

The identity of k_1 and k_2 with "partial multiplication constants" is thus made clear.

5. The Reflected Reactor

In order to proceed with the original problem we must derive and include the "contribution to A_2 due to reflector" term in (24). This is done as follows. Since by (10) k_1 = number of neutrons entering the fast group per neutron "absorbed" from the fast group, its reciprocal, $1/k_1$ = number of neutrons "absorbed" from the fast group per neutron entering the fast group, or which is the same thing, the probability of "absorption" of a group 1 neutron which was emitted in the core. Hence $1 - 1/k_1$ = the probability that a group 1 neutron will not be "absorbed" and hence will be absorbed in the (infinite) reflector. Since k_1A_1 = source of fast neutrons in the core,

$$(1 - 1/k_1)k_1A_1 = (k_1 - 1)A_1 = \text{sink of fast neutrons in the reflector,}$$

Now let

Σ_{12} = macroscopic cross section for conversion of a fast neutron to a slow neutron (same in reflector and core).

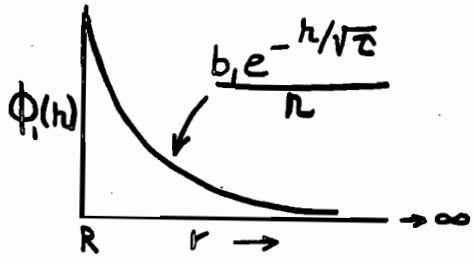
Σ_{1R} = total macroscopic "absorption" cross section for fast neutrons in the reflector.

P_{12} = probability of eventual absorption in the core of thermal neutrons born in the reflector from converted fast neutrons.

We can therefore write

$$(k_1 - 1)A_1 \frac{\Sigma_{12}}{\Sigma_{1R}} P_{12} = \text{contribution to } A_2 \text{ due to reflector} \tag{25}$$

We must derive an expression for P_{12}, which depends on the constants of group 2 and on the shape of the space distribution of group 1 neutrons in the reflector:



In order to do this we shall consider the following "two-group one-velocity" problem with the constants and true absorption of the thermal group, group 2. Let us call these groups "x" and "2" and define

C_x = absorption rate of group "x" neutrons in the core

C_2 = absorption rate of group "2" neutrons in the core

$\Sigma_{xC} = \Sigma_C$

$\Sigma_{xR} = \Sigma_R + \Sigma_{x2}$, temporarily considered arbitrary

Σ_{x2} = macroscopic cross section for conversion of group "x" neutrons into group "2" neutrons, $= \Sigma_{xR} - \Sigma_R$

$D_x = D$ (same in core and in reflector)

k_x = number of neutrons entering group "x" per neutron absorbed from group "x", also temporarily considered to be arbitrary.

In the core one absorption of either "x" group neutrons or "2" group neutrons must yield k_2 fresh neutrons for criticality. For convenience these will be assumed to enter the "x" group; <u>no absorptions</u> give rise to group "2" neutrons in this problem, all such neutrons being presumed to come only from conversion of group "x" neutrons into group "2" neutrons in the reflector; P_{x2} is the probability of eventual absorption in the core of these group "2" neutrons. By analogy with equations (23), (24) and (25) we must have the following balance:

$$C_x = \frac{1}{k_x}\left[S(x \rightarrow x)C_x + S(2 \rightarrow x)C_2\right] \tag{23'}$$

$$C_2 = \frac{1}{k_2}\left[S(x \rightarrow 2)C_x + S(2 \rightarrow 2)C_2\right] + (k_x - 1)C_x \frac{\Sigma_{x2}}{\Sigma_{xR}} \cdot P_{x2} \tag{24'}$$

where, by postulate,

$$S(x \rightarrow x) = S(2 \rightarrow x) = k_2$$
$$S(x \rightarrow 2) = S(2 \rightarrow 2) = 0.$$

Solving (23') and (24') under the condition that C_x and C_2 are not zero yields

$$P_{x2} = \frac{(k_x - k_2)\,\Sigma_{xR}}{k_2(k_x - 1)\,\Sigma_{x2}}$$

So far the quantities Σ_{xR} and k_x have been purposely left arbitrary. We shall now choose them in such a fashion that P_{x2} becomes identical with P_{12}, which will be the case if our choice is such that the space distribution in the reflector of "x" neutrons becomes identical with that of group 1 neutrons there in our original problem. This is achieved by choosing

$$\Sigma_{xR} = \frac{D}{\tau} \tag{26}$$

and

$$k_x = 1 + B_1^2 L_C^2 \tag{27}$$

Since $\Sigma_{xR} = \Sigma_R + \Sigma_{x2}$ and $\Sigma_R = \frac{D}{L^2}$ we now have

$$\Sigma_{x2} = D\left[\frac{1}{\tau} - \frac{1}{L^2}\right] \tag{28}$$

Using (26), (27), (28) and (19) in the expression for P_{x2} yields P_{12}:

$$P_{12} = \frac{(B_1^2 - B_2^2)}{(1 + B_2^2 L_C^2)\, B_1^2 \tau \left[\frac{1}{\tau} - \frac{1}{L^2}\right]} \tag{29}$$

Returning now to the absorption rate balance for the core for our original problem,

$$A_1 = \frac{1}{k_1}\left[S(1\rightarrow 1)A_1 + S(2\rightarrow 1)A_2\right] \tag{25}$$

$$A_2 = \frac{1}{k_2}\left[S(1\rightarrow 2)A_1 + S(2\rightarrow 2)A_2\right] + (k_1 - 1)A_1 P_{12}\frac{\Sigma_{12}}{\Sigma_{1R}} \tag{26}$$

The S's are given by (22). $\frac{\Sigma_{12}}{\Sigma_{1R}}$ is the resonance escape probability p, here taken to be practically unity. Using the condition that A_1 and A_2 are not both zero we have therefore

$$\begin{vmatrix} k_1 & -\eta f \\ 1 + k_2(k_1 - 1)P_{12} & -k_2 \end{vmatrix} = 0 \tag{31}$$

whence

$$\eta f = \frac{k_1 k_2}{1 + k_2(k_1 - 1)P_{12}} = \frac{(1 + B_1^2\tau)(1 + B_2^2 L_c^2)}{1 + \left[\dfrac{B_1^2 - B_2^2}{\dfrac{1}{\tau} - \dfrac{1}{L^2}}\right]} \tag{32}$$

Applying (7) and (8) and solving for C gives

$$C = \frac{1}{a}\left\{\frac{(1 + B_1^2\tau)(1 + B_2^2 L_c^2)}{\eta\left[1 + \dfrac{B_1^2 - B_2^2}{\dfrac{1}{\tau} - \dfrac{1}{L^2}}\right] - (1 + B_1^2\tau)}\right\} \tag{33}$$

Letting $B_1R = x_1$ and $B_2R = x_2$ we rewrite (14) and (18) as

$$\frac{\tan x_1}{x_1} = -\frac{\sqrt{\tau}}{R} \tag{14'}$$

$$\frac{\tan x_2}{x_2} = -\frac{L}{R} \tag{18'}$$

where R is either known or a value is assumed. Using the table of values of tanx/x from Jahnke and Emde, "Tables of Functions," we find x_1 and x_2 and then substitute x_1/R for B_1 and x_2/R for B_2 in (33). Recalling the definition of a from Paragraph 2, we write the final solution as

$$\boxed{C = \frac{A_u\sigma_m}{A_m\sigma_u}\left\{\frac{(R^2 + x_1^2\tau)(R^2 + x_2^2 L^2)}{\eta\left[R^2 + \left(\dfrac{x_1^2 - x_2^2}{L^2 - \tau}\right)L^2\tau\right] - (R^2 + x_1^2\tau)}\right\} \frac{\text{kg. fuel atoms}}{\text{liter sol'n}}} \tag{34}$$

If C were given and it were required to find R for criticality, solution can be effected quite readily by assuming various values of R and finding the corresponding concentrations C(R) required for criticality, then plotting C(R) versus R and reading the required R from the graph.

6. Numerical Example

Let it be required to find the concentration of U^{235} in a water-moderated, water-reflected homogeneous thermal reactor of core radius 9 inches = 22.8 cm. We have $L = 2.88$ cm.; $L^2 = 8.3$ cm^2; $\tau = 33$ cm^2; $\frac{A_u \sigma_m}{A_m \sigma_u} = \frac{235 \times 0.64}{18 \times 645} = 0.013$
$\eta = 2.11$

$$\frac{\tan x_1}{x_1} = -\frac{\sqrt{33}}{22.8} = -0.252; \text{ find } x_1 = 2.5672$$

$$\frac{\tan x_2}{x_2} = -\frac{2.88}{22.8} = -0.126; \text{ find } x_2 = 2.8022$$

Substituting these values in (34) gives $C \cong 0.0276 \frac{\text{kg. } U^{235}}{\text{liter}}$. (1.37 kg. total U^{235}). Allowing for the approximations and inherent uncertainties we would then expect criticality for a concentration of, say, 27.6 ± 2 grams/liter.

7. Extension of the Method

By making the appropriate changes in equations (14) and (18) and elsewhere as required, the restriction of equality of D for the core and the reflector may be relaxed and the method applied to cases of reflectors which differ from the core moderator. The method may also be applied to a multi-group analysis, but then becomes much more complex since quantities such as P_{13}, P_{23}, P_{123}, etc. must be calculated.

In applying the method to other than spherical geometries the appropriate buckling equations must be used in place of (14) and (18) or (14') and (18'). For example, with an infinitely long cylinder of radius R, (14') and (18') would become

$$\frac{x_1 J_1(x_1)}{J_o(x_1)} = \frac{R}{\sqrt{\tau}} \cdot \frac{K_1(R/\sqrt{\tau})}{K_o(R/\sqrt{\tau})} \quad \text{and} \quad \frac{x_2 J_1(x_2)}{J_o(x_2)} = \frac{R}{L} \cdot \frac{K_1(R/L)}{K_o(R/L)} .$$