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HEAT TRANSFER IN NONCIRCULAR DUCTS

PART I

H.C. CLAIBORNE

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HEAT TRANSFER IN NONCIRCULAR DUCTS

PART I

Laminar Flow Velocity Distributions and Heat Transfer Characteristics for Noncircular Ducts with Fully Developed Hydrodynamic and Thermal Boundary Layers

By

H. C. Claiborne

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SUMMARY

This is the first of a planned series of reports on the analytical treatment of heat transfer to noncircular ducts with emphasis on liquid metal systems. This report is concerned only with results for systems with fully developed hydrodynamic and thermal boundary layers; i.e., away from entrance effects.

Analytical solutions for velocity distributions in the case of laminar or viscous flow are presented for rectangular, equilateral triangular, right isosceles triangular, elliptical and circle sector ducts.

Analytical solutions are given for the temperature distribution in the case of slug flow in the type ducts mentioned above for heat transfer at constant wall temperature and at uniform wall heat flux. In addition, the general right triangular duct problem is solved for the case of uniform wall flux. The applicability of these results to liquid metal systems for a region of the turbulent regime is indicated.

At uniform wall flux the average Nussult modulus for slug flow was found to be 6, 4, 3, and 2 for rectangular, equilateral triangular, right isosceles triangular, and 30 degree right triangular ducts respectively.

INTRODUCTION

The subject of fluid flow and heat transfer in noncircular ducts from a fundamental viewpoint has been virtually neglected in the literature. This probably resulted from the industrial practice of generally using round pipes in heat transfer equipment. With the advent of nuclear engineering and the resulting unconventional heat transfer design problems and the increasing industrial use of noncircular ducts in heat exchangers, the problem becomes more than just an academic question.

In computing heat transfer coefficients based on the momentum transfer theory or other theories, it is necessary to know the velocity distribution. Nikuradse (ref. 9 and 10) made extensive measurements of the velocity distribution for turbulent flow in several noncircular ducts, but, unfortunately, these measurements are confined to the turbulent core. In addition, the development of the hydrodynamic relationships for turbulent flow in noncircular ducts is complicated by the existence of secondary flow in corners. This was established experimentally by Nikuradse (ref. 9). As yet no generalized velocity distribution relation has been established as in the case of flow in pipes.

Average heat transfer coefficients for rectangular ducts have been determined experimentally and correlated on the basis of the conventional equivalent diameter by Bailey and Cope (ref. 1) and by Washington and Marks (ref. 14).

Eckert and Low (ref. 4) developed a numerical method for the determination of the temperature distributions and the heat transfer characteristics of a heat exchanger composed of polygonal flow passages with the walls heated uniformly by internal sources.

The circumferential variation of the local heat transfer coefficient was estimated from Nikuradse's data by postulating similarity between the velocity and temperature fields; i.e., for fluids whose Pr = 1. Thus, as in the classical Reynolds Analogy, the influence of the laminar sub-layer is not considered and the results are not applicable to fluids whose Prandtl modulus differs appreciably from unity.

The general vein of this work is to attempt to arrive at analytical solutions for the various conditions resulting from the hydrodynamics, method of heat application and the geometry of the systems considered with particular emphasis on liquid metal systems. In this particular report, the heat transfer equations are developed for the case of slug flow in several noncircular ducts for conditions far downstream. It is shown that the slug solutions are applicable to liquid metal systems in the turbulent regime for relatively low Reynolds moduli.

The author wishes to express his appreciation to C. L. Perry for his assistance in some of the more difficult mathematical points and to H. F. Poppendiek for his criticisms and suggestions.

NOMENCLATURE

Any consistent set of units may be used.

a,	a duct dimension
A,	heat transfer area
b,	a duct dimension
B,	a constant
c,	heat capacity
C,	a constant
D_e,	equivalent diameter (4 x hydraulic radius)
E,	complete elliptic integral of the second kind
$E'(p)$,	a constant
f,	functional notation
$F'(p)$,	a constant
g,	an integer
h,	fraction of a duct wall dimension
h_c,	heat transfer coefficient
i,	$\sqrt{-1}$
k,	molecular thermal conductivity
K,	total thermal conductivity (eddy + molecular)
n,	an integer
N,	ratio of long side to short side of a rectangular duct
Nu,	Nusselt modulus
P,	pressure
p,	a constant
Pe,	Peclet modulus

Pr, Prandtl modulus

q, heat transferred per unit time

q_f, heat transferred per unit time per unit length of duct

r, coordinate line in polar coordinate system

R, a vector normal to heat transfer surface

Re, Reynolds modulus

s, an integer

t, temperature

t_c, minimum temperature of system

t_m, mixed-mean temperature of the fluid

t_w, inside duct wall temperature

t_{wm}, mean inside wall temperature

T, time

u, fluid velocity at any particular point

U, average or slug flow fluid velocity

v, fraction of a duct wall dimension

w, fraction of a duct wall dimension

x, coordinate axis in cartesian coordinate system

y, coordinate axis in cartesian coordinate system

z, coordinate axis in cartesian coordinate system (coincides with duct axis)

α, molecular diffusivity of heat, $\frac{k}{c\gamma}$

β, circle sector angle

Δ, a finite increment

ϵ, eddy diffusivity of heat

η, coordinate line in elliptic coordinate system

θ, coordinate in polar coordinate system or an acute angle of a right triangle

λ, separation constant

μ, viscosity

ν, kinematic viscosity

ξ, coordinate line in elliptic coordinate system

γ, fluid density

ϕ, functional notation

ψ, functional notation

∇^2, Laplacian operator

ANALYTICAL VELOCITY DISTRIBUTION SOLUTIONS FOR LAMINAR FLOW IN A DUCT

Equation of Motion

The motion of a fluid is generally represented by the Navier-Stokes equation (ref. 5). For the case of laminar flow considerable simplification occurs. A simpler and more direct way of obtaining the laminar flow equation is by a force balance on an elemental cube of fluid such as was done by Purday (ref. 11).

For steady state laminar flow of an incompressible fluid, consider an elemental cube of fluid of dimensions Δx, Δy, Δz. Take z in the direction of flow with x and y perpendicular to z. The difference in force on the two faces separated by a distance Δz is

$$\left(P + \frac{dP}{dz}\,\Delta z\right)\Delta x\,\Delta y - P\,\Delta x\,\Delta y = \frac{dP}{dz}\,\Delta x\,\Delta y\,\Delta z \tag{1}$$

The difference in viscous shear on the two faces normal to x is

$$\mu\left[\frac{\partial}{\partial x}\left(u + \frac{\partial u}{\partial x}\,\Delta x\right)\right]\Delta y\,\Delta z - \mu\,\frac{\partial u}{\partial x}\,\Delta y\,\Delta z = \mu\,\frac{\partial^2 u}{\partial x^2}\,\Delta x\,\Delta y\,\Delta z \tag{2}$$

and the difference in viscous shear on the two faces normal to y is

$$\mu\left[\frac{\partial}{\partial y}\left(u + \frac{\partial u}{\partial y}\,\Delta y\right)\right]\Delta x\,\Delta z - \mu\,\frac{\partial u}{\partial y}\,\Delta x\,\Delta z = \frac{\partial^2 u}{\partial y^2}\,\Delta x\,\Delta y\,\Delta z \tag{3}$$

Since the difference in total force on the two faces must balance the net shear on the four sides when no extraneous forces are present and $\frac{\partial u}{\partial z} = 0$ for incompressible fluid in laminar flow

$$\frac{dP}{dz}\,\Delta x\,\Delta y\,\Delta z = \mu\,\frac{\partial^2 u}{\partial x^2}\,\Delta x\,\Delta y\,\Delta z + \mu\,\frac{\partial^2 u}{\partial y^2}\,\Delta x\,\Delta y\,\Delta z \tag{4}$$

$$\frac{dP}{dz} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \nabla^2 u \tag{5}$$

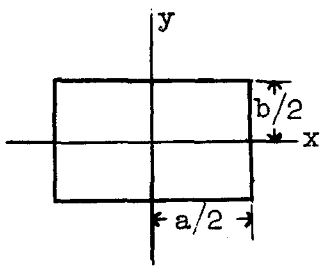
For constant isothermal flow in a uniform straight duct,

$$\frac{dP}{dz}\Big/\mu = \text{constant} = B \tag{6}$$

Therefore, $\nabla^2 u = B$ is the differential equation for the velocity (7)
distribution in ducts of any shape. The boundary conditions are given by $u = 0$ all along the boundary of the shape in question.

The equation and the boundary conditions for laminar flow in a duct are the same that occurs in the theory of elasticity when considering the torsion of beams. A number of these solutions is outlined by Love (ref. 6), by Sokolnikoff (ref. 12) and by Timoshenko (ref. 13). The complete solutions for laminar flow velocity distributions are shown in the following sections for several ducts of different geometry.

<u>Rectangular Duct</u>



$$\nabla^2 u = B \tag{7}$$

Boundary Conditions:

$$\left.\begin{aligned} u(\pm a/2, y) &= 0 \\ u(x, \pm b/2) &= 0 \end{aligned}\right| \tag{8}$$

$$\text{Choose } u = C_1 \left[\Psi(x,y) - \frac{1}{2}(x^2 + y^2)\right] \tag{9}$$

The problem will be solved if a $\Psi(x,y)$ can be found that is an even harmonic in the region bounded by $x = \pm a/2$, $y = \pm b/2$ and assumes on the boundaries the values of $\frac{1}{2}(x^2 + y^2)$. This follows from the differential equation.

For these conditions,

$$\Psi(x,y) = \frac{a^2}{4} + \frac{1}{2}(y^2 - x^2) - \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty} \frac{(-1)^n \cosh \frac{(2n+1)\pi y}{a}}{(2n+1)^3 \cosh \frac{(2n+1)\pi b}{2a}} \cos \frac{(2n+1)\pi x}{a} \tag{10}$$

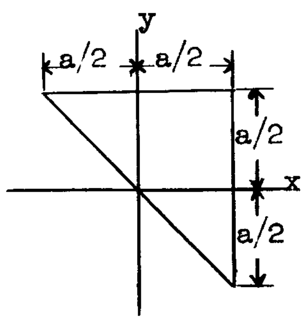
Therefore,

$$u = C_1\left[\frac{1}{2}\left(y^2 - \frac{a^2}{4}\right) + \frac{a^2}{4} - \frac{1}{2}(x^2 + y^2) - \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n \cosh \frac{(2n+1)\pi y}{a}}{(2n+1)^3 \cosh \frac{(2n+1)\pi b}{2a}} \cos \frac{(2n+1)\pi x}{a}\right] \tag{11}$$

From the differential equation and the assumed solution $C_1 = -\frac{k}{2}$; therefore,

$$u = \frac{B}{2}\left[x^2 - \frac{a^2}{4} + \frac{8\,a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n\cosh\frac{(2n+1)\pi y}{a}}{(2n+1)^3\cosh\frac{(2n+1)\pi b}{2a}}\cos\frac{(2n+1)\pi x}{a}\right] \tag{12}$$

Right Isosceles Triangular Duct



$$\nabla^2 u = B \tag{7}$$

$$\text{Boundary Conditions:}\quad \left.\begin{aligned} u(x,a/2) &= 0\\ u(a/2,y) &= 0\\ u(x,\ -x) &= 0\end{aligned}\right| \tag{13}$$

$$\text{Choose } u = C_1\left[\Psi(x,y) - \frac{1}{2}\ (x^2 + y^2)\right] \tag{14}$$

where $\Psi(x,y)$ is a harmonic function that reduces to $\frac{1}{2}(x^2 + y^2)$ on the boundaries.

From the differential equation and the assumed solution, $C_1 = \frac{k}{2}$.

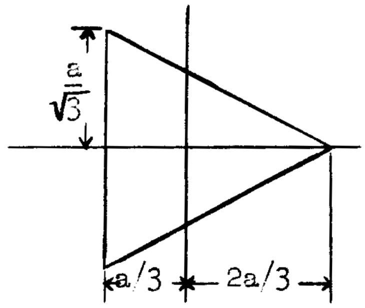
From the solution for the rectangular cross section the following harmonic function that satisfies the boundary conditions can be constructed.

$$\Psi(xy) = -xy + \frac{a}{2}(x+y) - \frac{4a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3\sinh\frac{(2n+1)\pi}{2}}\left[\sinh\frac{(2n+1)\pi y}{a}\cos\frac{(2n+1)\pi x}{a} + \sinh\frac{(2n+1)\pi x}{a}\cos\frac{(2n+1)\pi y}{a}\right] \tag{15}$$

Therefore the solution is

$$u = \frac{B}{2}\left\{\frac{1}{2}(x^2 + y^2) + xy - \frac{a}{2}(x+y) + \frac{4a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^3\sinh\frac{(2n+1)\pi}{2}}\left[\sinh\frac{(2n+1)\pi y}{a}\cos\frac{(2n+1)\pi x}{a} + \sinh\frac{(2n+1)\pi x}{a}\cos\frac{(2n+1)\pi y}{a}\right]\right\} \tag{16}$$

<u>Equilateral Triangular Duct</u>



$$\nabla^2 u = B \tag{7}$$

$$\text{Boundary Conditions: } \left.\begin{aligned} u(-a/3,y) &= 0 \\ u\left(x,\frac{-x}{\sqrt{3}} + \frac{2a}{3\sqrt{3}}\right) &= 0 \\ u\left(x,\frac{x}{\sqrt{3}} - \frac{2a}{3\sqrt{3}}\right) &= 0 \end{aligned}\right| \tag{17}$$

The problem is the same type as before but the solution is in the form of polynomials.

$$\text{Choose } \Psi(x,y) = C_2\,(x + iy)^n + C_3 \text{ where } n = 3 \tag{18}$$

$$\text{The real part of } C_2(x + iy)^3 + C_3 = C_2(x^3 - 3\,xy^2) + C_3 \tag{19}$$

$$\text{Therefore, } u = C_1\left[C_2\,(x^3 - 3\,xy^2) - \frac{1}{2}\,(x^2 + y^2) + C_3\right] \tag{20}$$

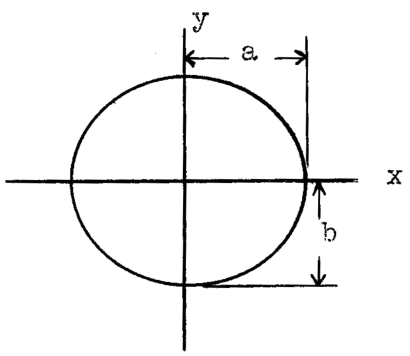
If C_1, C_2 and C_3 are evaluated by use of the boundary conditions and the original differential equation,

$$u = \frac{B}{2}\left[\frac{1}{2}\,(x^2 + y^2) - \frac{1}{2a}\,(x^3 - 3\,xy^2) - \frac{2a^2}{27}\right] \tag{21}$$

Expressing in the more convenient form of the product of the three equations for the sides of the triangle as shown,

$$u = -\frac{B}{4a}\,(x - 3\,y - \frac{2}{3}\,a)(x + 3\,y - \frac{2a}{3})\,(x + \frac{a}{3}) \tag{22}$$

<u>Elliptical Duct</u>



$$\nabla^2 u = B \tag{7}$$

$$\text{Boundary Condition: } u(x, b^2 - \frac{b^2}{a^2}\,x^2) = 0 \tag{23}$$

As in the preceeding section, choose $\Psi(x,y) = C_2(x+iy)^n + C_3$ where $n = 0$ (24)

$$\text{The real part of } C_2 (x+iy)^2 = C_2 (x^2 - y^2) \tag{25}$$

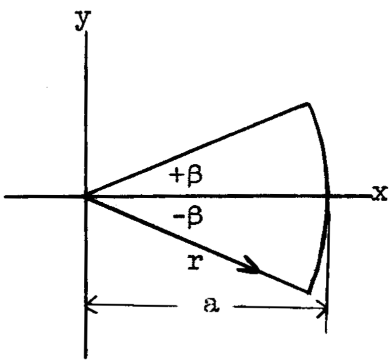
$$\text{Therefore,} \quad u = C_1 \left[C_2(x^2 - y^2) + C_3 - \frac{1}{2} (x^2 + y^2) \right] \tag{26}$$

If C_1, C_2 and C_3 are evaluated by use of the boundary conditions and the original differential equation,

$$u = \frac{B}{4} \left[x^2 + y^2 - \frac{a^2 - b^2}{a^2 + b^2} (x^2 - y^2) - \frac{2\, a^2 b^2}{a^2 + b^2} \right] \tag{27}$$

<u>Circle Sector Duct</u>

$$\nabla^2 u = B \tag{7}$$



This problem is easier to solve when expressed in polar coordinates.

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = B \tag{28}$$

$$\text{Boundary Conditions:} \quad u(r, \pm \beta) = 0, \quad u(a, \pm \theta) = 0 \tag{29}$$

$$\text{Assume,} \quad u = C_1 \left[\Psi(r,\theta) - \frac{1}{2} r^2 \right] \tag{30}$$

$$\text{Therefore,} \quad C_1 = - \frac{B}{2} \tag{31}$$

The harmonic function is found to be

$$\Psi(r,\theta) = \frac{1}{2} r^2 \frac{\cos 2\theta}{\cos 2\beta} + a^2 \sum_{n=0}^{\infty} C_{2n+1} \left(\frac{r}{a}\right)^{\frac{(2n+1)\pi}{2\beta}} \cos (an+1) \frac{\pi\theta}{2\beta} \tag{32}$$

where

$$C_{2n+1} = (-1)^{n+1} \left[\frac{1}{(2n+1)\pi - 4\beta} - \frac{2}{(2n+1)\pi} + \frac{1}{(2n+1)\pi + 4\beta} \right] \tag{33}$$

therefore,

$$u = B \left[\frac{1}{4} r^2 \left(1 - \frac{\cos 2\theta}{\cos 2\beta}\right) - \frac{a^2}{2} \sum_{n=0}^{\infty} C_{2n+1} \left(\frac{r}{a}\right)^{(2n+1)\frac{\pi}{2\beta}} \cos (2n+1) \frac{\pi\theta}{2\beta} \right] \tag{34}$$

General Heat Transfer Equation

The general equation for the transfer of heat to a fluid flowing in a duct is

$$\frac{\partial}{\partial x}\left(K\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial t}{\partial z}\right)=c\gamma\left(u_x\frac{\partial t}{\partial x}+u_y\frac{\partial t}{\partial y}+u_z\frac{\partial t}{\partial z}+\frac{\partial t}{\partial T}\right) \tag{35}$$

where K is the total conductivity (molecular + eddy) and T is the time. For steady state conditions the time term, $\frac{\partial t}{\partial T}$, is zero, so the equation to be solved reduces to

$$\frac{\partial}{\partial x}\left(K\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial t}{\partial z}\right)=c\gamma\left(u_x\frac{\partial t}{\partial x}+u_y\frac{\partial t}{\partial y}+u_z\frac{\partial t}{\partial z}\right) \tag{36}$$

If the heat capacity, c, and the density, γ, can be considered independent of temperature and consequently independent of the coordinate system,

$$\frac{\partial}{\partial x}\left[(\alpha+\epsilon)\frac{\partial t}{\partial x}\right]+\frac{\partial}{\partial y}\left[(\alpha+\epsilon)\frac{\partial t}{\partial y}\right]+\frac{\partial}{\partial z}\left[(\alpha+\epsilon)\frac{\partial t}{\partial z}\right]=c\gamma\left(u_x\frac{\partial t}{\partial x}+u_y\frac{\partial t}{\partial y}+u_z\frac{\partial t}{\partial z}\right) \tag{37}$$

where α is the molecular diffusivity of heat and ϵ is the eddy diffusivity of heat. For a fluid flowing in a duct, the only net velocity component is along the axis, z, of the duct. Therefore,

$$\frac{\partial}{\partial x}\left[(\alpha+\epsilon)\frac{\partial t}{\partial x}\right]+\frac{\partial}{\partial y}\left[(\alpha+\epsilon)\frac{\partial t}{\partial y}\right]+\frac{\partial}{\partial z}\left[(\alpha+\epsilon)\frac{\partial t}{\partial z}\right]=c\gamma\frac{\partial t}{\partial z} \tag{38}$$

Solving the resulting heat transfer equation depends on the complexity of the functions representing ϵ and u. For the case of turbulent flow in pipes it is possible to obtain analytical solutions to the heat transfer equation by using approximate functions for u and ϵ derived from experimental data. For the case of turbulent flow in noncircular ducts, insufficient experimental data are extant for approximating the necessary functions. In addition, the differential equation is more complex because the velocity, u, is a function of two coordinates instead of one as for pipes.

Modification of Heat Transfer Equation for Liquid Metals

Postulate that some average value for ϵ and u may be used and that α is independent of the temperature. With these simplifications equation (38) becomes

$$(\alpha + \epsilon_{av}) \nabla^2 t = U \frac{\partial t}{\partial z} \tag{39}$$

The question now arises as to whether the simplified equation approximates reality for any system. It is evident that the equation exactly represents a system that has Re = ∞ and Pr = 0 provided that as Re approaches infinity, the value of ϵ does not approach infinity. From a more practical viewpoint the equation should represent the liquid metal systems for relatively low values of Reynolds modulus in the turbulent regime since Pr $\sim$ 0.01 for most liquid metals. Actually slug flow cannot exist; however, a square wave representation (slug flow) for the velocity should be a fair approximation because the temperature gradient near the wall is not as large for liquid metals as it is for other fluids.

A partial experimental verification of the postulate is furnished by the Lyon equation (ref. 7) for liquid metals in turbulent flow through pipes for uniform wall heat flux. Examination of the Lyon equation, $Nu = 7 + 0.025 (Pe)^{0.8}$, reveals that the Nusselt modulus is approximately constant for relatively low values of the Peclet modulus. For Pe = 100 (say for Re = 10,000, Pr = 0.01), Nu = 8, which is exactly the value for slug flow (square velocity wave) in round pipes with no eddy diffusion. When Pe = 1000, the term containing the Peclet modulus, which roughly may be considered as the eddy diffusion contribution, becomes 47 percent of the total value of the Nusselt modulus. Therefore, it appears that a slug flow solution approximately represents a liquid metal system when the Peclet modulus is around 100. For higher values of the Peclet modulus, an experimental value for the eddy diffusion, ϵ_{av}, would be needed for better correlation.

The argument presented above is shown graphically by Figure 1. The eddy diffusivity of heat was considered equal to the diffusivity of momentum which was computed according to the classical manner. The ratio, $\frac{\alpha + \epsilon}{\nu} = \frac{1}{Pr} + \frac{\epsilon}{\nu}$, was used instead of $(\alpha + \epsilon)$ to make the results independent of the individual physical properties. It is seen from the figure that for liquid metals, $(\epsilon + \alpha)$ is primarily composed of α for relatively low Reynolds moduli and using an average value of ϵ would lead to only a small error. For other fluids the opposite is the case.

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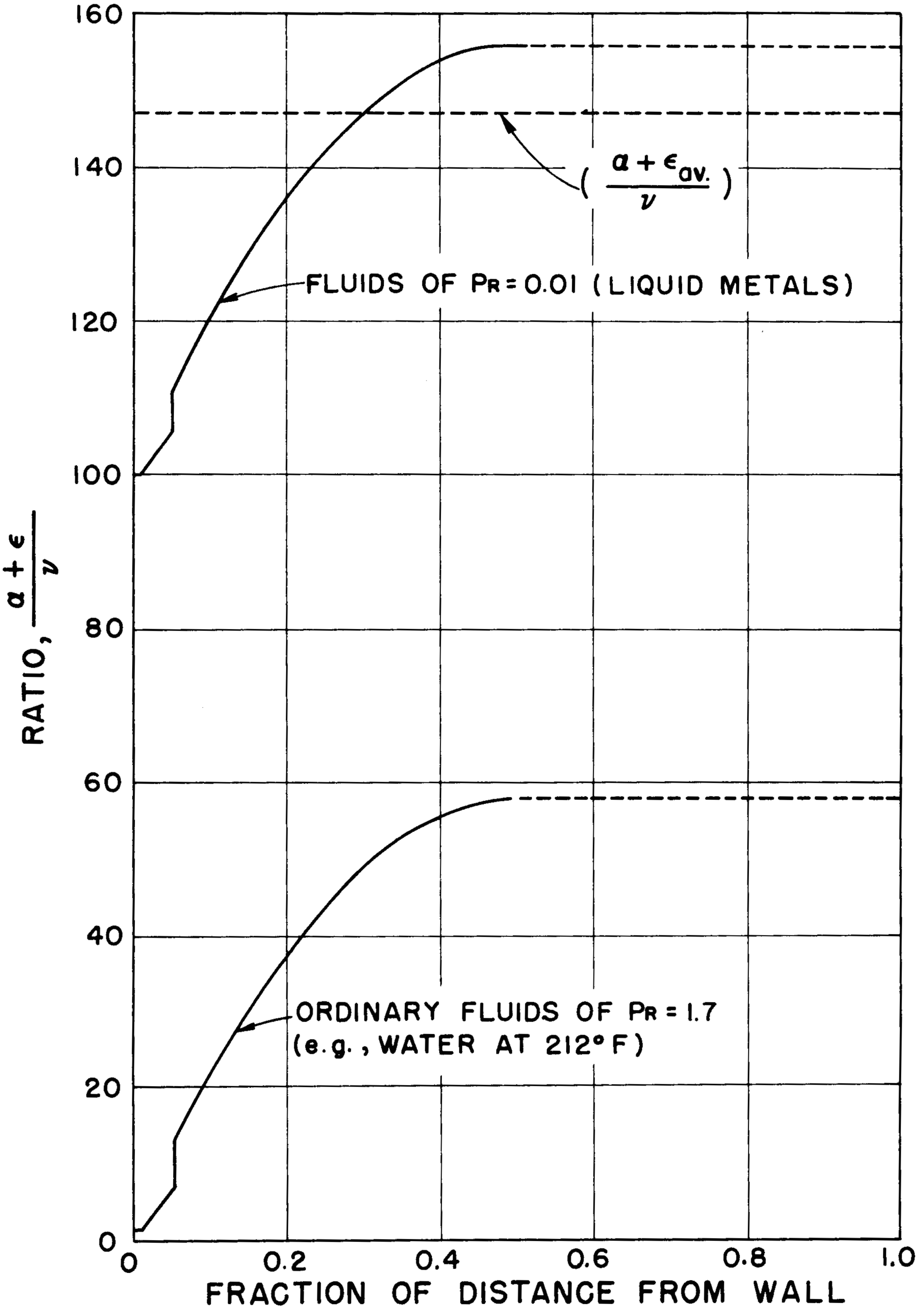


RATIO OF TOTAL DIFFUSITY OF HEAT TO KINEMATIC VISCOSITY IN PIPES FOR Re = 20,000

FIG. 1

ANALYTICAL SOLUTIONS OF THE HEAT TRANSFER EQUATION FOR SLUG FLOW IN NONCIRCULAR DUCTS

Constant Wall Temperature

In the theoretical treatment of heat transfer to fluids in ducts the two limiting cases - constant wall temperature and constant flux - are usually considered. The case of constant wall temperature in pipes means that the wall temperature is constant along the length of pipe. The temperature around the circumference is constant because the flux is applied uniformly around the circumference and the heat flows unidirectionally along the radii. In the case of a noncircular duct, application of a uniform heat flux around the periphery cannot produce a constant wall temperature around the periphery because the heat flow is not unidirectional. In maintaining a constant wall temperature around a noncircular duct, a nonuniform flux would occur around the periphery.

In order to solve equation (6) for the case of constant wall temperature, the customary simplifying assumptions are made. The temperature gradient, $\partial t/\partial z$, is not exactly constant for this case; however, when dealing with conditions for downstream, the change of $\frac{\partial t}{\partial z}$ with length is very small. Consequently the rate of change of the temperature gradient, $\partial^2 t/\partial z^2$, along the length of the duct is negligible compared with the rates of change of the other gradients ($\partial^2 t/\partial x^2$ and $\partial^2 t/\partial y^2$). Therefore,

$$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{c\ \gamma\ U}{K}\left(\frac{\Delta t}{\Delta z}\right)_{av.} \qquad (40)$$

For any shape the temperature around the wall is constant and equal to t_w. Thus the form of the equation and boundary conditions are identical with that for the laminar velocity distribution in noncircular ducts and the solutions are identical. For this case $B = \frac{c\ \gamma\ U}{K}\left(\frac{\Delta t}{\Delta z}\right)_{av.}$ and u is replaced by $(t - t_w)$.

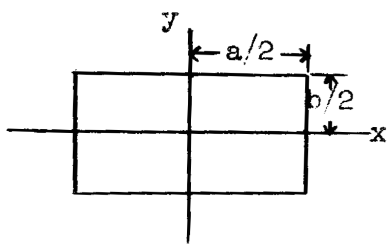
Constant Wall Flux

For this case the temperature gradient along the axis, $\frac{\partial t}{\partial z}$, far downstream is constant and $\partial t/\partial z = dt/dz$. Consequently $\partial^2 t/\partial z^2 = 0$ and

$$\nabla^2 t = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = \frac{c\,\gamma\,U}{K}\left(\frac{dt}{dz}\right) \tag{41}$$

The ease of solution now depends on the boundary conditions. For flux = q/A anywhere along the wall fluid interface, $\frac{dt}{dR} = -\frac{q}{KA}$ where R is the normal to the wall. This follows from the Fourier conduction law.

Rectangular Duct



The total heat transferred per unit time is obviously the sum of the heat transferred through the four sides. By a heat balance

$$ab\,U\,\gamma\,c\,dt = 2Kb\,dz\left(\frac{\partial t}{\partial x}\right)_{\pm a/2,y} + 2\,Ka\,dz\left(\frac{\partial t}{\partial y}\right)_{x,\pm\frac{b}{2}} \tag{42}$$

Therefore,

$$\frac{c\,\gamma\,U}{K}\left(\frac{dt}{dz}\right) = \frac{2(a+b)}{ab}\;\frac{q}{KA} \tag{43}$$

and the differential equation (41) becomes

$$\nabla^2 t = \frac{2(a+b)}{ab}\;\frac{q}{KA}$$

Boundary conditions:

$$\left(\frac{\partial t}{\partial x}\right)_{\pm\frac{a}{2},\,y} = \left(\frac{\partial t}{\partial y}\right)_{x,\,\pm b/2} = \pm\frac{q}{KA}$$

$$(t - t_c)_{0,0} = 0 \tag{44}$$

where t_c is the minimum temperature of the system. It is not necessary to use the minimum temperature of the system, any point of known temperature would suffice. Selection of any other point would only add a constant to the equation.

The form of a solution is $t - t_c = C_1 x^2 + C_2 y^2$ (45)

Applying the first condition gives $C_1 = \frac{q}{a\ KA}$ and $C_2 = \frac{q}{b\ KA}$

Since the assumed form produces the correct conditions at the boundary and satisfies the differential equation,

$$t - t_c = \left(\frac{x^2}{a} + \frac{y^2}{b} \right) \frac{q}{KA} \tag{46}$$

Let v = fraction of a/2 from center to corner, w = fraction of b/2 from center to corner, and $N = a/b$, then substituting into equation (46)

$$t_w - t_c = \frac{a\ q}{4\ KA} \left(v^2 + \frac{w^2}{N} \right) \tag{47}$$

Since $\mathbf{A} = 2(a+b)\Delta z$, equation (47) becomes (48)

$$\frac{t_w - t_c}{q_f\ /\ K} = \frac{1}{8(N+1)} \left(N\ v^2 + w^2 \right) \tag{49}$$

To illustrate the variation of the temperature along the walls, equation (49) for various values of N is shown plotted in Figure 2. It is easily seen that as the ratio of the sides, N, increases, the ratio of maximum temperature to minimum temperature along the wall increases.

For computing an average Nusselt modulus, define

$$Nu_{av} = \frac{h_c\ \ D_e}{k} \tag{50}$$

and

$$h_c = \frac{q}{A(t_{wm} - t_m)} \tag{51}$$

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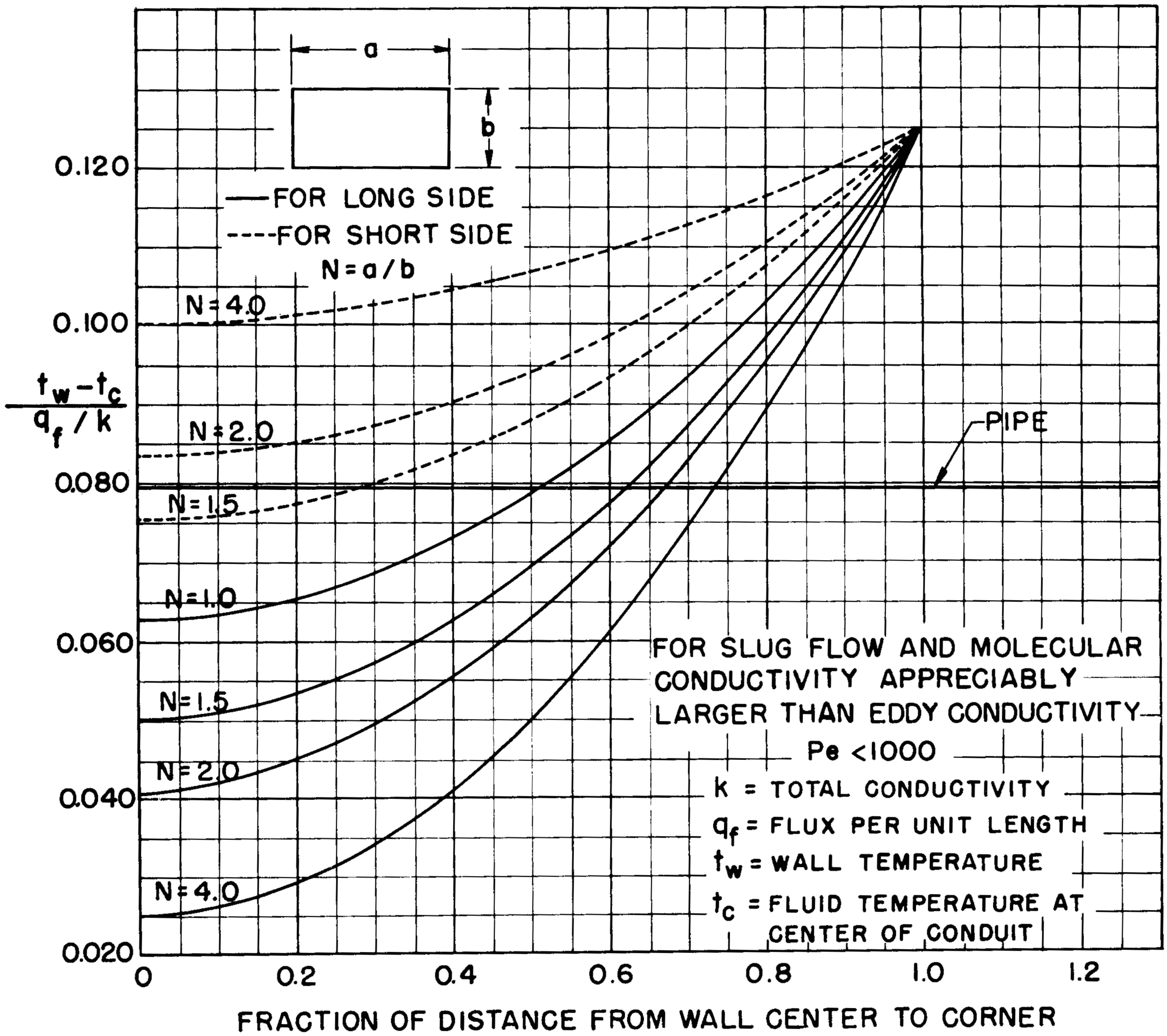


FIGURE 2

TEMPERATURE DISTRIBUTION ALONG WALLS OF RECTANGULAR CONDUIT FOR UNIFORM WALL FLUX.

where t_{wm} is the mean wall temperature and t_m is the mixed mean fluid temperature.

$$t_{wm} - t_m = \frac{\int_{-a/2}^{a/2} \frac{q}{KA}\left[\left(x^2 + \frac{b}{4}\right) + t_c\right] dx + \int_{-b/2}^{b/2} \frac{q}{KA}\left[\left(\frac{a}{4} + \frac{y^2}{4}\right) + t_c\right] dy}{a + b}$$

$$\frac{\int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \left[\frac{q}{KA} \; \frac{x^2}{a} + \frac{y^2}{b} + t_c\right] dx\, dy}{ab} \qquad (52)$$

Integrating and collecting terms,

$$t_{wm} - t_m = \frac{q}{KA} \; \frac{ab}{3(a + b)} \qquad (53)$$

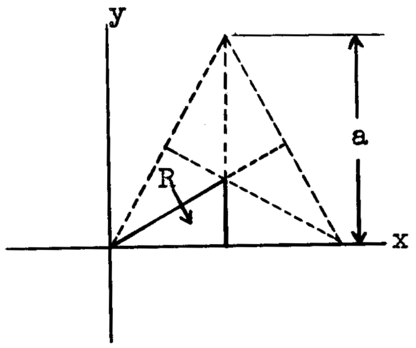
The equivalent diameter (4 x hydraulic radius), $D_e = \frac{2\ ab}{a + b}$ (54)

Substituting equations (51), (53) and (54) into equation (50),

$$\frac{k}{K} \; Nu_{av.} = 6 \qquad (55)$$

Thus the Nusselt modulus is independent of the ratio of the two sides. Note that for slug flow with no eddy diffusion, K = k. For the liquid metal system the value of K must be determined experimentally for relatively high values of the Peclet modulus.

Equilateral Triangular Duct



No heat can flow across the medians; therefore, it is only necessary to consider one-sixth of the complete section as shown in the accompanying figure.

A heat balance gives,

$$6\,U\,\frac{a}{6}\,\frac{a}{\sqrt{3}}\,\gamma c\,dt = -6\,K\,\frac{a}{3}\,dz\left(\frac{\partial t}{\partial y}\right)_{x,0} \tag{56}$$

Therefore,

$$\frac{c\,\gamma}{K}\,U\left(\frac{dt}{dz}\right) = \frac{6}{a}\,\frac{q}{KA} \tag{57}$$

and the differential equation becomes

$$\nabla^2 t = \frac{6}{a}\,\frac{q}{KA} \tag{58}$$

Boundary conditions:

$$\left.\begin{aligned}
&\left(\frac{dt}{dR}\right)_{x,\;x/\sqrt{3}} = 0\\
&\left(\frac{\partial t}{\partial x}\right)_{a/\sqrt{3},\;y} = 0\\
-&\left(\frac{\partial t}{\partial y}\right)_{x,0} = \frac{q}{KA}\\
&(t - t_c)_{a/\sqrt{3},\,a/3} = 0
\end{aligned}\right| \tag{59}$$

To put the first boundary condition in usable form, the following relationship for the normal derivative is used.

$$\frac{dt}{dR} = \frac{\partial t}{\partial x}\frac{dx}{dR} + \frac{\partial t}{\partial y}\frac{dy}{dR} \tag{60}$$

Therefore,

$$\left(\frac{dt}{dR}\right)_{x,x/\sqrt{3}} = \left(\frac{\partial t}{\partial x}\right)_{x,x/\sqrt{3}} + \left(\frac{\partial t}{\partial y}\right)_{x,x/\sqrt{3}} = \frac{1}{2}\left(\frac{\partial t}{\partial x}\right)_{x,x/\sqrt{3}} - \frac{\sqrt{3}}{2}\left(\frac{\partial t}{\partial y}\right)_{x,x/\sqrt{3}} = 0 \tag{61}$$

The form of a solution is

$$t - t_c = C_1 (x^2 + y^2) + C_3 x + C_4 y + C_5 \tag{62}$$

Applying the boundary conditions and solving for the constants, the following solution is obtained which satisfies both the differential equation and the boundary conditions.

$$t - t_c = \left[\frac{3}{2a} (x^2 + y^2) - \sqrt{3}\, x - y + \frac{2}{3} a\right] q/KA \tag{63}$$

Since this is a symetrical case, the temperature distribution for each wall is equivalent; so only the wall at $y = 0$ is considered. Therefore,

$$t_w - t_c = \left[\frac{3}{2a} x^2 - \sqrt{3}\, x + \frac{2}{3} a\right] q/KA \tag{64}$$

Let
$$x = \frac{a}{\sqrt{3}} (1-v) \tag{65}$$

Substituting equation (65) into (64), $t_w - t_c = (3 v^2 + 1) \dfrac{qa}{6\,KA}$ (66)

Since
$$A = 2 \sqrt{3}\, a\, \Delta z \tag{67}$$

$$\frac{t_w - t_c}{q_f/K} = \frac{3 v^2 + 1}{12 \sqrt{3}} \tag{68}$$

Equation (68) is shown plotted in Figure 3.

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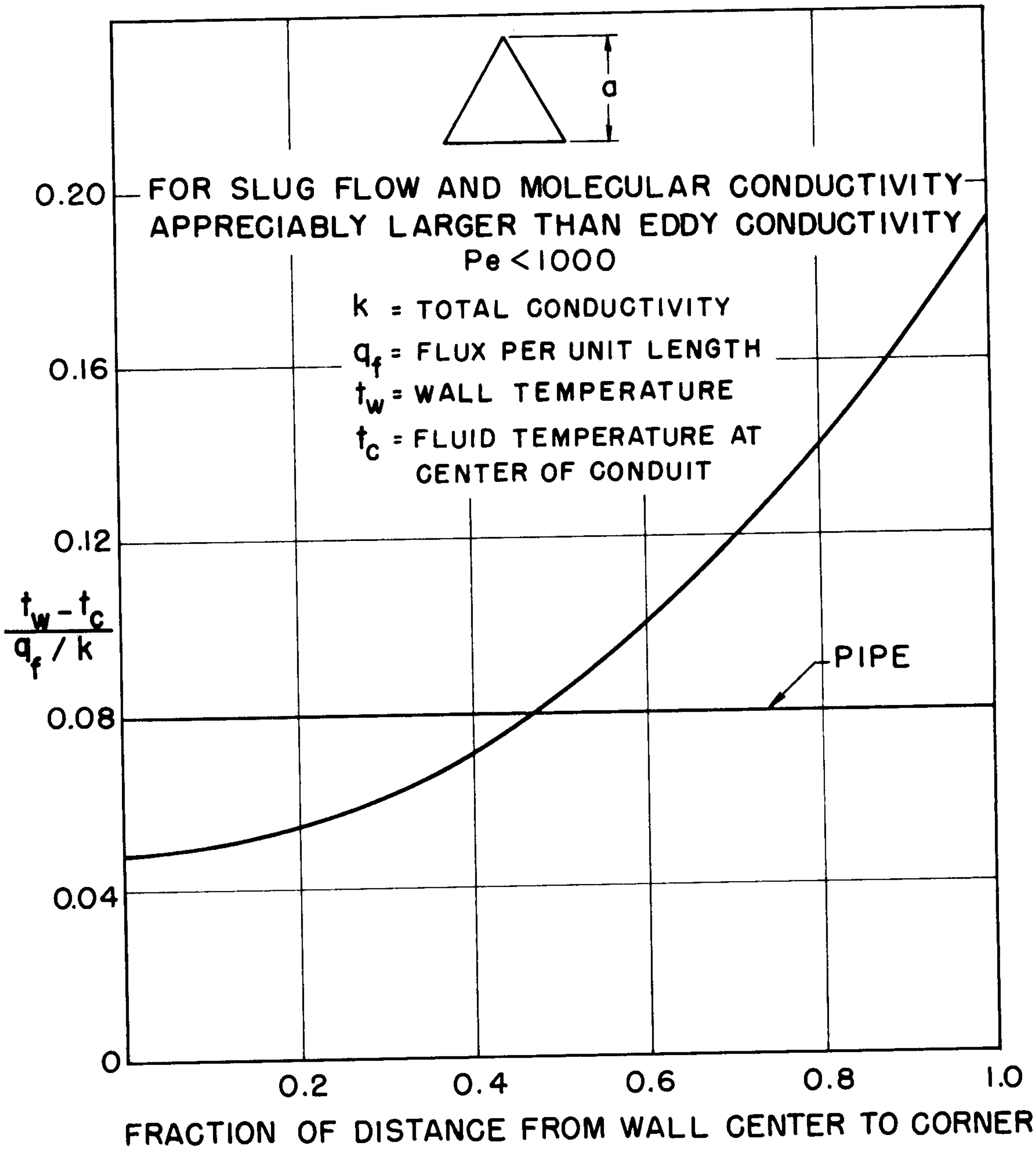


FIGURE 3
TEMPERATURE DISTRIBUTION ALONG WALLS OF EQUILATERAL TRIANGULAR DUCT FOR UNIFORM WALL FLUX.

$$t_{wm} - t_m = \frac{\sqrt{3}}{2a} \int_0^{\frac{2a}{\sqrt{3}}} \left[\frac{q}{KA} \left(\frac{3}{2} x^2 - \sqrt{3}x + \frac{2}{3} a \right) + t_c \right] dx -$$

$$\frac{6\sqrt{3}}{a^2} \int_0^{\frac{a}{3}} \int_{\sqrt{3}y}^{\frac{a}{\sqrt{3}}} \left\{ \frac{q}{KA} \left[\frac{3}{2a} (x^2 + y^2) - \sqrt{3}x - y + \frac{2}{3} a \right] + t_c \right\} dx\, dy \quad (69)$$

Integrating and simplifying,

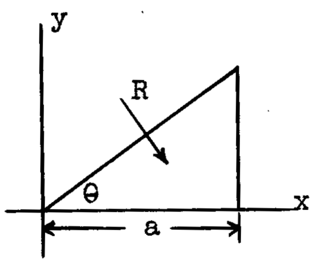
$$t_{wm} - t_m = \frac{aq}{3KA} + t_c - \frac{a}{6} \frac{q}{KA} - t_c = \frac{a}{6} \frac{q}{KA} \quad (70)$$

$$D_e = \frac{2a}{3} \quad (71)$$

Substituting into equation (50),

$$\frac{k}{K} Nu_{av} = 4 \quad (72)$$

Right Triangular Duct



For the general right triangle as shown in the accompanying figure, a heat balance gives

$$U \frac{a^2}{2} \tan \theta\, \gamma c\, dt = -Ka \sec \theta\, dz \left(\frac{\partial t}{\partial R}\right)_{x,x \tan \theta} - Ka\, dz \left(\frac{\partial t}{\partial y}\right)_{x,0} + Ka\, dz \left(\frac{\partial t}{\partial x}\right)_{a,y}$$

(73)

Therefore , $$\nabla^2 t = \frac{c\,\gamma}{K} U \left(\frac{dt}{dz}\right) = \frac{2}{a} \frac{q}{KA} \left(\csc\theta + \operatorname{ctn}\theta + 1\right) \tag{74}$$

Boundary conditions:

$$\left(\frac{dt}{dR}\right)_{x,\ x\tan\theta} = -\frac{q}{KA}\ \text{, or applying equation (60)}$$

$$\left.\begin{aligned} &\left(\frac{\partial t}{\partial x}\right)_{x,\ x\tan\theta} \sin\theta - \left(\frac{\partial t}{\partial y}\right)_{x,x\tan\theta} \cos\theta = -\frac{q}{KA} \\ &\left(\frac{\partial t}{\partial y}\right)_{x,\ 0} = -\frac{q}{KA} \\ &\left(\frac{\partial t}{\partial x}\right)_{a,\ y} = \frac{q}{KA} \\ &(t - t_c) = 0 \text{ where } \frac{\partial t}{\partial x} = \frac{\partial t}{\partial y} = 0 \end{aligned}\right| \tag{75}$$

The form of a solution is

$$t - t_c = C_1\,(x^2 + y^2) + C_2\,x + C_3\,y + C_4 \tag{76}$$

Using the boundary conditions to evaluate the constants the following solution that satisfies all conditions is obtained.

$$t-t_c = \frac{q}{KA}\left[\frac{1}{2a}(1+\operatorname{ctn}\theta + \csc\theta)(x^2+y^2) - (\operatorname{ctn}\theta+\csc\theta)\,x - y + \frac{a}{2}\ \frac{1+(\operatorname{ctn}\theta+\csc\theta)^2}{1+(\operatorname{ctn}\theta+\csc\theta)}\right] \tag{77}$$

Let v, w, h equal the fraction of the distance between corners along ths side adjacent to θ, along the side opposite θ, and along the hypotenuse respectively.

Then, $x = v, a, y = wa \tan \theta$, and along the hypotenuse $x = ha$ (78)

$$A = a\,(1 + \sec \theta + \tan \theta)\, \Delta z \tag{79}$$

The following equation giving the wall temperatures along the side adjacent to θ when $w = 0$ and along the side opposite θ when $v = 1$ is obtained by substituting equations (78) and (79) into (77).

$$\frac{t_w - t_c}{q_f/K} = \frac{\operatorname{ctn} \theta}{2}\,(v^2 + w^2 \tan^2 \theta) - \frac{1}{1 + \tan \theta + \sec \theta}\left[(\operatorname{ctn} \theta + \csc \theta)\, v + w \tan \theta - \frac{1 + (\operatorname{ctn} \theta + \csc \theta)^2}{2(1 + \operatorname{ctn} \theta + \csc \theta)}\right] \tag{80}$$

Similarly the temperature distribution along the hypotenuse is found to be

$$\frac{t_w - t_c}{q_f/K} = \frac{h^2}{\sin 2\theta} - \frac{1}{1 + \tan \theta + \sec \theta}\left[(\operatorname{ctn} \theta + \csc \theta + \tan \theta)\, h - \frac{1 + (\operatorname{ctn} \theta + \csc \theta)^2}{2\,(1 + \operatorname{ctn} \theta + \csc \theta)}\right] \tag{81}$$

Equations (80) and (81) are shown plotted in Figure 4 for several values of θ. The plot clearly shows that temperature variation along a wall becomes greater as the angle θ decreases.

For evaluating the average Nusselt modulus, the mean temperatures are computed as follows:

$$t_m = \frac{2}{a^2 \tan \theta} \int_0^a \int_0^{x \tan \theta} \frac{q}{KA}\left[\frac{1}{2a}(1 + \operatorname{ctn} \theta + \csc \theta)(x^2+y^2) - (\operatorname{ctn} \theta + \csc \theta)x - y + \frac{a}{2}\,\frac{1 + (\operatorname{ctn} \theta + \csc \theta)^2}{1 + \operatorname{ctn} \theta + \csc \theta} + t_c\right] dx\, dy \tag{82}$$

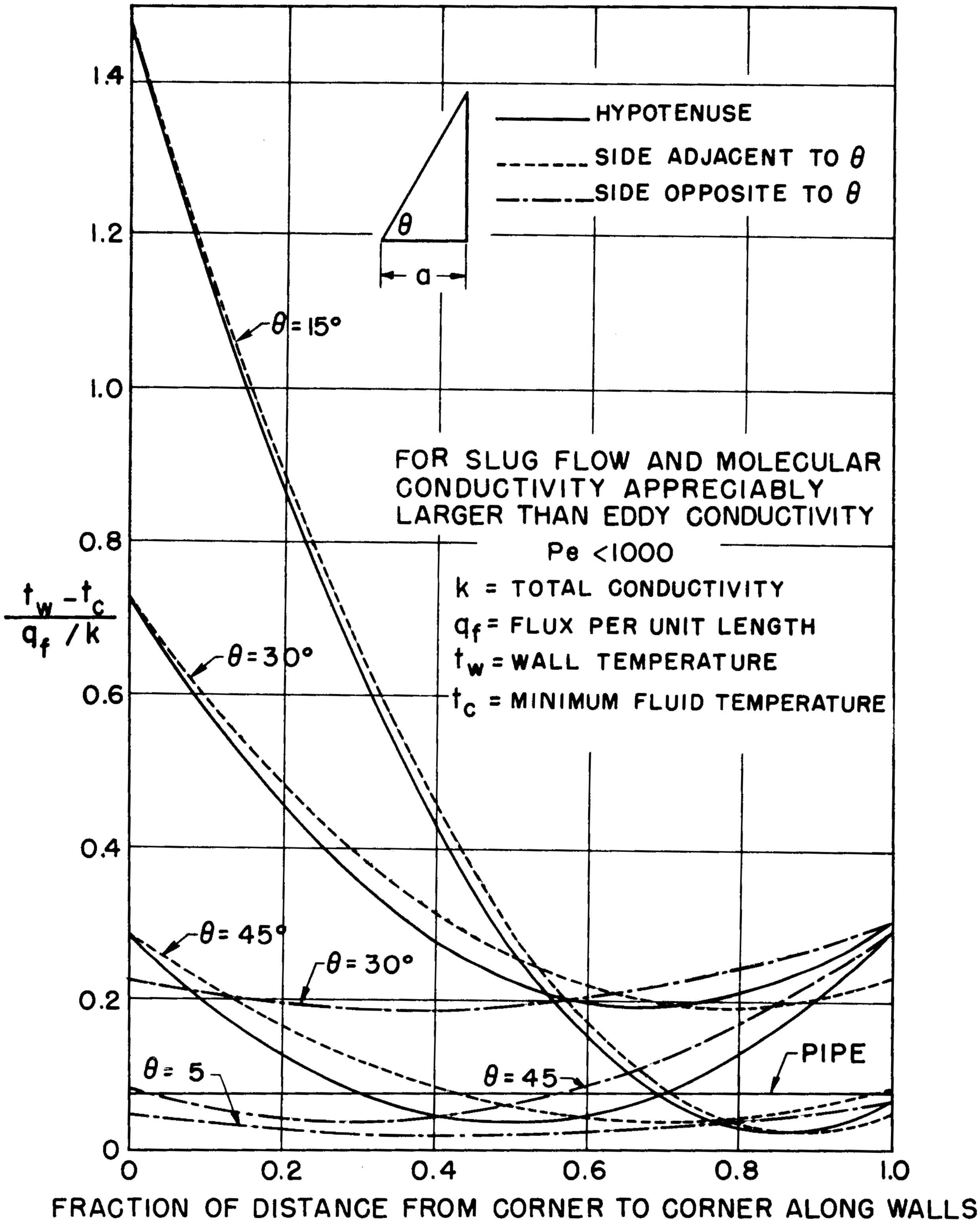


FIGURE 4
TEMPERATURE DISTRIBUTION ALONG WALLS OF RIGHT TRIANGULAR DUCT FOR UNIFORM WALL FLUX

Integrating,

$$t_m = \frac{qa}{KA}\left[\frac{1}{4}(1 + \text{ctn}\,\theta + \csc\theta)(1 + \frac{\tan^2\theta}{3}) - \frac{2}{3}(\text{ctn}\,\theta + \csc\theta + \frac{\tan\theta}{2}) + \frac{1 + (\text{ctn}\,\theta + \csc\theta)^2}{2\,(1 + \text{ctn}\,\theta + \csc\theta)}\right] + t_c \tag{83}$$

$$t_{wm} = \frac{q/KA}{(1 + \tan\theta + h\sec\theta)}\left\{\int_0^a\left[\frac{1}{2h}(1 + \text{ctn}\,\theta + \csc\theta)x^2 - (\text{ctn}\,\theta + \csc\theta)\,x + \frac{h + h\,(\text{ctn}\,\theta + \csc\theta)^2}{2(1 + \text{ctn}\,\theta + \csc\theta)}\right]dx + a\,t_c + \int_0^{a\tan\theta}\left[\frac{1}{2a}(1 + \text{ctn}\,\theta + \csc\theta)(h^2+y^2) - (\text{ctn}\,\theta + \csc\theta)\,h - y + \frac{a + a\,(\text{ctn}\,\theta + \csc\theta)^2}{2\,(1 + \text{ctn}\,\theta + \csc\theta)}\right]dy + a\tan\theta\,t_c + \int_0^a\left[\frac{1}{2a}(1 + \text{ctn}\,\theta + \csc\theta)(1 + \tan^2\theta)x^2 - (\text{ctn}\,\theta + \csc\theta)x - x\tan\theta + \frac{a + a(\text{ctn}\,\theta + \csc\theta)^2}{2(1 + \text{ctn}\,\theta + \csc\theta)}\right]dx + a\sec\theta\,t_c\right\} \tag{84}$$

Integrating and combining terms,

$$t_{wm} = \frac{qa}{KA}\left\{\text{ctn}\,\theta\left[\frac{1}{6} + \frac{1}{2}\tan\theta(1 + \frac{\tan^2\theta}{3}) + \frac{\sec^3\theta}{6}\right] - \left[\frac{\text{ctn}\,\theta}{2} + \csc\theta + 1 + \frac{\tan^2\theta}{2} + \frac{1}{2}\sec\theta\,(\csc\theta + \tan\theta + 2)\right]\frac{1}{(1 + \tan\theta + \sec\theta)} + \frac{1 + \text{ctn}\,\theta\csc\theta}{1 + \text{ctn}\,\theta\csc\theta}\right\} + t_c \tag{85}$$

For this duct, $D_e = \dfrac{2a\tan\theta}{1 + \tan\theta + \sec\theta}$ (86)

Substituting equations (81), (82) and (83) into equation (50), which defines the Nusselt modulus,

$$\frac{k}{K} Nu_{av} = \frac{2 \tan \theta}{f(\theta)\ (1 + \tan \theta + \sec \theta)} \tag{87}$$

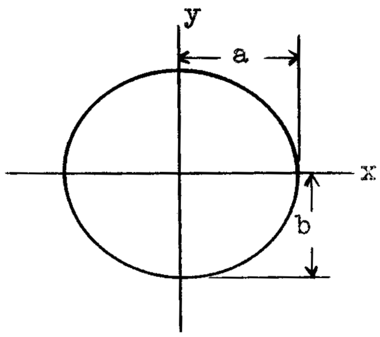
where

$$f(\theta) = \operatorname{ctn} \theta \left[\frac{1}{6} + \frac{1}{2} \tan \theta (1 + \frac{\tan^2 \theta}{3}) + \frac{\sec^3 \theta}{6}\right] - \left[\frac{\operatorname{ctn} \theta}{2} + \csc \theta + 1 + \frac{\tan^2 \theta}{2} + \frac{1}{2} \sec \theta \,(\csc \theta + \tan \theta + 2)\right] \left(\frac{1}{1 + \tan \theta + \sec \theta}\right) - \frac{1}{4}\left[1 + \operatorname{ctn} \theta + \csc \theta\right] \frac{(1 + \tan^2 \theta)}{3} + \frac{2}{3}\left(\operatorname{ctn} \theta + \csc \theta + \frac{\tan \theta}{2}\right) \tag{88}$$

The following table shows how the average Nusselt modulus varies with the angle θ.

θ	Nu_{av}
45°	3
30°	2
15°	0.3353
1°	0.000386

Elliptical Duct



The length of the perimeter of the ellipse $= 4a\, E\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right)$ where $E\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right)$ is a complete elliptic integral of the second kind. As shown in the figure, R is a vector always orthogonal to the periphery of the elliptical duct. The rate of heat transfer is given by

$$q = ab\,\pi\, U\, \gamma c\, dt = -4\, aK\, dz \left(\frac{\partial t}{\partial R}\right)_{x,\ f(x)} E\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right) \tag{89}$$

Therefore,

$$\nabla^2 t = \frac{c\gamma}{k} U \left(\frac{dt}{dz}\right) = - \frac{4aE\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right)}{ab\ \pi} \left(\frac{\partial t}{\partial R}\right)_{x,f(x)} = \frac{4E\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right)}{b\pi} \frac{q}{KA} \qquad (90)$$

To simplify the boundary conditions the cartesian coordinates are transformed to elliptic coordinates. The transformation (ref. 8) is made by substituting the following equations into equation (86).

$$x = \sqrt{a^2 - b^2} \cosh \xi \ \cos \eta \qquad (91)$$

$$y = \sqrt{a^2 - b^2} \sinh \xi \ \sin \eta \qquad (92)$$

Thus,

$$\frac{\dfrac{\partial^2 t}{\partial \xi^2} + \dfrac{\partial^2 t}{\partial \eta^2}}{(a^2 - b^2)(\cosh^2 \xi - \cos^2 \eta)} = \frac{4E\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right)}{b\pi} \frac{q}{KA} \qquad (93)$$

Boundary conditions:

$$\left.\begin{aligned} (t - t_c)_{\xi = 0, \eta = \frac{\pi}{2}} &= 0 \\ \left(\frac{dt}{dR}\right)_{x,f(x)} &= - \frac{q}{KA} \end{aligned}\right| \qquad (94)$$

The second boundary condition is transformed by differentiating equations (91) and (92) and substituting the result into the following equation which is always true for cartesian coordinates.

$$(dR)^2 = (dx)^2 + (dy)^2 = (a^2 - b^2)(\cosh^2 \xi - \cos^2 \eta)(d\xi^2 + d\eta^2) \qquad (95)$$

On a coordinate line where η = constant, $d\eta = 0$ and

$$- dR = \sqrt{a^2 - b^2} \sqrt{\cosh^2 \xi - \cos^2 \eta} \; d\xi \tag{96}$$

At the boundary

$$\xi = \xi_o , \quad \xi_o = \frac{1}{2} \ln \frac{a + b}{a - b} , \text{ and}$$

$$- dR = \sqrt{a^2 - b^2} \sqrt{\cosh^2 \xi_o - \cos^2 \eta} \tag{97}$$

Substituting this relation into equation (90),

$$\left(\frac{dt}{d\xi}\right)_{0,y} = \frac{q}{KA} \sqrt{a^2 - b^2} \sqrt{\cosh^2 \xi_o - \cos^2 \eta} \text{ which is} \tag{98}$$

the second boundary condition expressed in elliptic coordinates.

Let the solution be represented in the following form:

$$t - t_c = \left[\Psi(\xi,\eta) + \phi(\xi,\eta) \right] \frac{q}{KA} \tag{99}$$

where $\Psi(\xi,\eta)$ is a harmonic function even in ξ,η and $\phi(\xi,\eta)$ is a particular solution that satisfies the differential equation.

One may take $$\phi(\xi,\eta) = (\cosh^2 \xi + \cos^2 \eta) \frac{(a^2 - b^2) \, E\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right)}{b\pi} \tag{100}$$

and $$\Psi(\xi,\eta) = \sum_{n=0}^{\infty} C_n \cosh n\xi \cos n\eta \tag{101}$$

where the constants, C_n , are determined by the boundary conditions.

Applying the second boundary condition given by equation (98),

$$\sqrt{a^2 - b^2}\ \sqrt{\cosh^2 \xi_o - \cos^2\eta} = \frac{(a^2 - b^2)}{b\pi} E\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right) \sinh 2\xi_o + \sum_{n=0}^{\infty} n\, C_n \sinh n\xi_o \cos n\eta \tag{102}$$

The left side of equation (102) must be expanded into a Fourier series of cosines in order to evaluate C_n. Therefore,

$$\sqrt{a^2 - b^2}\ \sqrt{\cosh^2 \xi_o - \cos^2\eta} = \frac{a_o}{2} + a_1 \cos\eta + a_2 \cos 2\eta + \cdots \tag{103}$$

$$\text{where } a_n = \frac{4\sqrt{a^2 - b^2}}{\pi} \int_0^{\pi/2} \sqrt{\cosh^2 \xi_o - \cos^2\eta}\ \cos n\eta \ d\eta \tag{104}$$

From the orthogonality of the trigonometric functions, it follows that $a_n = 0$ when n is odd.

Now let $p = \dfrac{1}{\cosh \xi_o}$, $\eta = \dfrac{\pi}{2} - x$, and $n = 2n$, then

$$a_{2n} = \frac{4(-1)^n\sqrt{a^2 - b^2}}{p\pi} \int_0^{\frac{\pi}{2}} \sqrt{1 - p^2 \sin^2 x}\ \cos 2nx \, dx \tag{105}$$

From reference (2), $a_o = \dfrac{4\sqrt{a^2 - b^2}}{p\pi} E'(p)$ where $E'(p)$ is a constant. (106)

Comparing the coefficients of equations (102) and (103),

$$\frac{2\sqrt{a^2 - b^2}\ E'(p)}{p\pi} = \frac{(a^2 - b^2)}{b\pi}\ E\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right) \sinh 2\xi_o \ . \tag{107}$$

Therefore,

$$E'(p) = \frac{\sqrt{a^2 - b^2}}{b}\ E\left(\sqrt{1 - \left(\frac{b}{a}\right)^2}\right) \sinh \xi_o \tag{108}$$

Since $\cos 2x = 2 \cos^2 x - 1$, (109)

$$a_2 = -\ \frac{8\sqrt{a^2 - b^2}}{p\pi} \int_0^{\frac{\pi}{2}} \sqrt{1 - p^2 \sin^2 x}\ \cos^2 x\ dx + a_o \tag{110}$$

Therefore, (ref. 2),

$$a_2 = -\ \frac{8\sqrt{a^2 - b^2}}{3p^3\pi}\left[(1 + p^2)\ E'(p) - (1 - p^2)\ F'(p)\right] + a_o \tag{111}$$

$F'(p)$ is a constant. Comparing coefficients,

$$C_2 = \frac{1}{2 \sinh 2\xi_o}\left\{-\ \frac{8\ a^2 - b^2}{3p^3\pi}\left[(1 + p^2)\ E'(p) - (1 - p^2)\ F'(p)\right] + a_o\right\} \tag{112}$$

Since $\cos 4x = 8 \cos^4 x - 4 \cos 2x - 3$, (113)

$$a_4 = \frac{32\sqrt{a^2 - b^2}}{p\pi} \int_0^{\frac{\pi}{2}} \sqrt{1 - p^2 \sin^2 x}\ \cos^4 x\ dx + 4a_2 - 3a_o \tag{114}$$

Therefore (ref. 2),

$$a_4 = \frac{32\sqrt{a^2 - b^2}}{p^5\pi}\left[2(1 - 3p^2)F'(p) - (2 - 7p^2 - 3p^4)E'(p)\right] + 4a_2 - 3a_o \tag{115}$$

Comparing coefficients, $C_4 = \dfrac{a_4}{4 \sinh 4\xi_o}$ (116)

Since $\cos 6x = 32 \cos^6 x - 6 \cos 4x - 15 \cos 2x - 10$ (117)

$$a_6 = - \frac{128 \sqrt{a^2 - b^2}}{p\pi} \int_0^{\frac{\pi}{2}} \sqrt{1 - p^2 \sin^2 x} \cos^6 x \, dx + 6 a_4 - 15 a_2 + 10 a_o \qquad (118)$$

Therefore (ref. 2),

$$a_6 = - \frac{128 \quad a^2 - b^2}{105 \, p^7 \pi} \left[(8 - 33p^2 + 58p^4 + 15p^6)E'(p) - (8 - 29p^2 + 45p^2)(1-p^2)F'(p) \right] + 6a_4 - 15a_2 + 10a_o \qquad (119)$$

Comparing coefficients,

$$C_6 = \frac{a_6}{6 \sinh 6 \xi_o} \qquad (120)$$

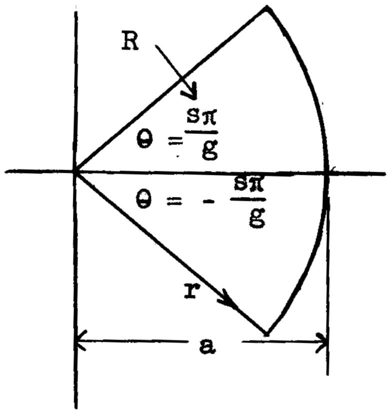
This process can be continued for as many constants as desired. Thus, the solution becomes

$$t - t_c = \left[(\cosh^2 \xi_o + \cos^2 \eta) \; \frac{(a^2 - b^2)}{b\pi} E \quad 1 - \frac{b}{a} \;^2 + C_o + \frac{a_2 \cosh 2\xi \cos 2\eta}{2 \sinh 2\xi_o} + \frac{a_4 \cosh 4\xi \cos 4\eta}{4 \sinh 4\xi_o} + \frac{a_6 \cosh 6\xi \cos 6\eta}{6 \sinh 6\xi_o} + \ldots \right] \frac{q}{KA} \qquad (121)$$

The constant $F'(p)$ is contained within the a_{2n}'s. A method of evaluation of $F'(p)$ is given in reference (3). Since $F'(p)$ is a function of the duct dimensions, $F'(p)$ must be evaluated for each particular case. After $F'(p)$ is obtained C_o is evaluated by the application of the first boundary condition.

<u>Circle Sector Duct</u>

This solution is a simplified version of the more elegant solution by Whitcombe (ref. 15) for a 60 degree sector. Whitcombe also demonstrated how the solution for the 60 degree sector approximates that for the equilateral triangle.



Since the cross sectional area of the sector is $a^2\Theta$ and the length of the arc subtended by 2Θ is $2a\Theta$, the heat transferred in a small length of the duct is

$$q = U\,a^2\;\frac{s}{g}\,\pi\gamma c\;dt = -\,2\,Ka\,dz\left(\frac{dt}{dR}\right)_{r,\,\frac{s}{g}\pi} + 2\,Ka\,\frac{s}{g}\,\pi\,dz\left(\frac{dt}{dr}\right)_{a,\Theta} \qquad (122)$$

Therefore,

$$\frac{C\,\gamma\,U}{K}\left(\frac{dt}{dz}\right) = \frac{2}{a}\left(\frac{g}{\pi s} + 1\right)\frac{q}{KA} \qquad (123)$$

Substituting into equation (50) and transforming to polar coordinates, the differential equation becomes

$$\frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\,\frac{\partial t}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 t}{\partial \Theta^2} = \frac{2}{a}\left(\frac{g}{\pi s} + 1\right)\frac{q}{KA} \qquad (124)$$

where g and s are integers.

The boundary conditions in polar coordinates are:

$$\left(\frac{dt}{dR}\right)_{r,\,\frac{s}{g}\pi} = -\,\frac{q}{KA} \quad \text{which becomes} \quad \left(\frac{\partial t}{\partial \Theta}\right)_{r,\,\frac{s\pi}{g}} = \frac{q}{KA}\,r$$

$$\left(\frac{\partial t}{\partial r}\right)_{a,\Theta} = \frac{q}{KA} \qquad (125)$$

$$(t - t_c) = 0 \text{ at } \Theta = 0 \text{ and at } r \text{ where } \frac{\partial t}{\partial r} = 0$$

A form of the solution is

$$t - t_c = \Big[\psi(r, \theta) + \phi(r)\Big] \frac{q}{KA} \tag{126}$$

where $\psi(r, \theta)$ is an even harmomic function and $\phi(r)$ is a particular integral that satisfies the differential equation.

Based on the boundary conditions, select

$$\phi(r) = \left(\frac{g}{\pi s} + 1\right) \frac{r^2}{2a} \tag{127}$$

and

$$\psi(r,\theta) = B\, r \cos\theta + \sum_{\lambda=0}^{\infty} C_\lambda\, r^\lambda \operatorname{Cos} \lambda\theta \tag{128}$$

Applying the first boundary condition,

$$\left(\frac{\partial t}{\partial \theta}\right)_{r,\, \frac{s}{g}\pi} = \frac{qr}{KA} = \left[-Br \sin \frac{s\pi}{g} - \sum_{\lambda=0}^{\infty} C_\lambda\, r^\lambda \sin \frac{s\lambda\pi}{g}\right] \frac{q}{KA} \tag{129}$$

Since λ is an arbitrary separation constant, take

$$\lambda = \frac{s}{g}\, n \text{ where } n = 1, 2, 3, \ldots$$

Therefore, from equation (129)

$$B = -\frac{1}{\sin \frac{s\pi}{g}} \tag{130}$$

Applying the second boundary condition,

$$\left(\frac{\partial t}{\partial r}\right)_{a,\theta} = \frac{q}{KA} = \left[\frac{q}{\pi s} + 1 - \frac{\operatorname{Cos}\theta}{\sin\frac{s\pi}{g}} + \sum_{n=0}^{\infty} \frac{gn}{s}\, C_\lambda\, a^{\left(\frac{gn}{s} - 1\right)} \operatorname{Cos} \frac{gn\theta}{s}\right] q/KA \tag{131}$$

Simplifying and considering the infinite series from n = 1 since the first term is zero when n = 0 (C_o is arbitrary).

$$1 - \frac{g}{\pi s} + 1 + \frac{\text{Cos } \theta}{\sin \frac{g\pi}{s}} = \sum_{n=1}^{\infty} \frac{gn}{s} C_\lambda \, a^{\left(\frac{gn}{s} - 1\right)} \text{Cos } \frac{gn\,\theta}{s} \tag{132}$$

Performing a Fourier expansion,

$$\int_{\frac{-s\pi}{g}}^{\frac{s\pi}{g}} \left[1 - \left(\frac{g}{\pi s} + 1\right) + \frac{\text{Cos } \theta}{\sin \frac{s\pi}{g}}\right] \text{Cos } \frac{gn\theta}{s} \, d\theta = \frac{gn}{s} \int_{\frac{-s\pi}{g}}^{\frac{s\pi}{g}} C_\lambda \, a^{\left(\frac{gn}{s} - 1\right)} \text{Cos}^2 \frac{gn\theta}{s} \, d\theta \tag{133}$$

Evaluating the integrals,

$$n\pi \, C_\lambda \, a^{\left(\frac{gn}{s} - 1\right)} = \frac{1}{\sin \frac{s\pi}{g}} \left\{ \frac{\sin\left[\frac{s\pi}{g}\left(1 + \frac{gn}{s}\right)\right]}{\frac{gn}{s} + 1} + \frac{\sin\left[\frac{s\pi}{g}\left(\frac{gn}{s} - 1\right)\right]}{\frac{gn}{s} - 1} \right\} \tag{134}$$

Simplifying and solving for C_λ ,

$$C_\lambda = \frac{1}{n\pi \sin \frac{s\pi}{g} \, a^{\left(\frac{gn}{s} - 1\right)}} \quad \frac{(1)^{n+1}}{\left(\frac{gn}{s}\right)^2 - 1} \tag{135}$$

Therefore,

$$t - t_c = \left[C_o + \left(\frac{g}{s\pi} + 1\right) \frac{r^2}{2a} - \frac{r \text{ Cos } \theta}{\sin \frac{s\pi}{g}} + \frac{a}{\pi \sin \frac{s\pi}{g}} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{r}{a}\right)^{\frac{gn}{s}} \text{Cos } \frac{gn\theta}{s}\right] q/KA \tag{136}$$

The arbitrary constant, C_o, is determined by applying the third boundary condition to equation (136). First solving for r where $\theta = 0$ and $\frac{\partial t}{\partial r} = 0$,

$$0 = \left(\frac{g}{s\pi} + 1\right) r - \frac{1}{\sin \frac{s\pi}{g}} + \frac{ag}{s\pi \sin \frac{s\pi}{g}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \left(\frac{r}{a}\right)^{\left(\frac{gn}{s} - 1\right)}}{\left(\frac{gn}{s}\right)^2 - 1} \tag{137}$$

Equation (137) cannot be solved explicitly for r; it must be solved by approximate methods for the particular values of a, s and g in order to evaluate C_o from equation (136).

FUTURE WORK

The analytical study of heat transfer to noncircular duct is a long range program that will be continued as time permits. Other more difficult problems under consideration in the order listed are:

1. Solution of the heat transfer equation for fully developed viscous flow.
2. Entrance solutions for slug flow.
3. Some solutions of the heat transfer equation for irregular annuli such as for flow parallel to tube banks.
4. Solution of the heat transfer equation for fully developed turbulent flow using approximate relations for velocity distribution.
5. Heat transfer through noncircular ducts with thick walls.
6. Approximate entrance solutions for turbulent and viscous flow.

HCC:gd

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