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# CRITICAL SIZE AND BREEDING GAIN OF RESONANCE PILES

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NOTE: (The following report was written in winter of 1946; its publication was delayed because the authors left the project before it was completed. It is being published now nevertheless because there are several important methods of calculation described in it. A series of numerical calculations, mainly on Beryllium systems, have been carried out to illustrate the principles described here. Normally these numerical results would have been included as part of this report. However, since this was found to be unfeasible, we simply give the following list of monthly reports in which the numerical computations are summarized):

- (1) Mon S-124, pgs. 42-43
- (2) CF-3490, Part I, pgs. 30-32
- (3) CF-3352

A.M.W.

## I. Critical Size

(a) We shall consider the problem of critical size and breeding gain calculations on the basis of the theory developed by Friedman and Monk in CF-2881. If the Fermi theory of slowing down is applicable, the negative Laplacian  $\nabla^2$  is determined by

$$1 = \int_0^{\infty} \eta(\tau) \int_{\tau}^{\infty} e^{-(\tau - \tau')/\lambda} f(\tau') \frac{dP(\tau', \tau)}{d\tau'} d\tau' d\tau \quad (1a)$$

where  $\tau$  is the "age" of the neutrons (taken to be zero at thermal energies),  $\eta(\tau)$  is the number of neutrons produced



per absorption of any kind at age  $\tau$ ,  $f(\tau)$  is the normalized fission spectrum ( $\int_0^\infty f(\tau) d\tau = 1$ ),  $P(\tau', \tau)$  is the probability that a neutron of age  $\tau'$  escape resonance capture in slowing down to age  $\tau$ . We have assumed that all neutrons are captured before becoming thermal. (cf. CF-2331, eq. 5,  $P(\tau', 0) = 0$ .)

To solve (1) for  $\bar{\kappa}^2$ , we expand the exponential and integrate termwise. In order to get better convergence, we first take out the factor  $e^{-\frac{\tau'}{\bar{\tau}} \bar{\kappa}^2}$  where  $\bar{\tau}$  is a mean age from fission to capture, averaged over reproduction. We shall see how to choose  $\bar{\tau}$  shortly. We are left then with

$$1 = e^{-\frac{\tau'}{\bar{\tau}} \bar{\kappa}^2} \int_0^\infty \eta(\tau) \int_\tau^\infty \left[ 1 + \left\{ \frac{\tau'}{\bar{\tau}} - (\tau' - \tau) \right\} \bar{\kappa}^2 + \frac{\left\{ \frac{\tau'}{\bar{\tau}} - (\tau' - \tau) \right\}^2 \bar{\kappa}^4}{2} \dots \right] f(\tau') \frac{dP(\tau', \tau)}{d\tau} d\tau' d\tau.$$

It is reasonable to choose  $\bar{\tau}$  in such a way that on integrating termwise, the second term vanishes. We also note that if the reactor is infinite, we may define an average multiplication constant  $\bar{k}$ :

$$\bar{k} = \int_0^\infty \eta(\tau) \int_\tau^\infty f(\tau') \frac{dP(\tau', \tau)}{d\tau} d\tau' d\tau. \quad (3a)$$

Our  $\bar{\tau}$ , chosen as indicated above is given by

$$\bar{\tau} = \frac{\int_0^\infty \eta(\tau) \int_\tau^\infty (\tau' - \tau) f(\tau') \frac{dP(\tau', \tau)}{d\tau} d\tau' d\tau}{\bar{k}}. \quad (4a)$$

We also define  $\bar{\tau}^2$ , the mean square age, similarly:

$$\bar{\tau}^2 = \frac{\int_0^{\infty} \eta(\tau) \int_{\tau}^{\infty} (\tau' - \tau)^2 f(\tau') \frac{dP(\tau', \tau)}{d\tau} d\tau' d\tau}{\bar{k}} \quad (5a)$$

With these definitions we may write eq. (3) as

$$1 = e^{-\bar{\tau} \lambda_0^2} \lambda_0^2 \left[ \bar{k} + 0 + \frac{\lambda_0^4 \bar{k}}{2} (\bar{\tau}^2 - \bar{\tau}^2) + \dots \right] \quad (6a)$$

If  $\bar{\tau}^2 - \bar{\tau}^2$  is not too large, we may take for our first approximation

$$e^{-\bar{\tau} \lambda_0^2} = \bar{k} \quad \text{or} \quad \lambda_0^2 = \frac{\ln \bar{k}}{\bar{\tau}}$$

A better approximation will be

$$\lambda_1^2 = \frac{\ln \bar{k}}{\bar{\tau}} + \frac{\ln [1 - (\ln \bar{k})^2 \beta]}{\bar{\tau}}$$

with

$$\beta = \frac{1}{2} \left( \frac{\bar{\tau}^2}{\bar{\tau}^2} - 1 \right)$$

Since we expect  $\beta$  to be small, we may use

$$\lambda_1^2 = \frac{\ln \bar{k}}{\bar{\tau}} [1 + \beta \ln \bar{k}] = \lambda_0^2 [1 + \beta \ln \bar{k}] \quad (7a)$$

We note that the first approximation  $\lambda_0^2$  is the result one would get if  $\tau$  were the exact age from fission to capture:

for example, if there were a single fission energy and a single capture energy. The  $\bar{\tau}$  that is actually used is an average over the ages from fission to capture for the various possible ways a neutron may go in the actual system. In this average the  $\eta$  at capture is a weight function.  $\bar{\tau}$  includes the "premature capture correction" considered by Weinberg in CF-2848 X (Eq. 6). We shall discuss this relationship more fully in the next section. The correction term  $\beta \lambda n \bar{k}$  arises from the higher moments of the fission to capture path distribution. It corresponds to the spectral corrections considered in CF-2831 (p. 15).

(b) We shall now give the results of the preceding section under the simplifying assumption that there exists a definite fission energy; i.e.,  $f(\tau) = \delta(\tau - \tau_f)$ . The equation corresponding to eq. (1a) of section (a) will be numbered (1b), etc.

$$1 = \int_0^{\tau_f} \eta(\tau) e^{-(\tau - \tau_f)/\bar{\tau}} \frac{dP(\tau_f, \tau)}{d\tau} d\tau \quad (1b)$$

$$\bar{k} = \int_0^{\tau_f} \eta(\tau) \frac{dP(\tau_f, \tau)}{d\tau} d\tau \quad (3b)$$

$$\bar{\tau} = \frac{\int_0^{\tau_f} \eta(\tau) (\tau_f - \tau) \frac{dP(\tau_f, \tau)}{d\tau} d\tau}{\bar{k}} \quad (4b)$$

$$\bar{\tau}^2 = \frac{\int_0^{\tau_f} \eta(\tau) (\tau_f - \tau)^2 \frac{dP(\tau_f, \tau)}{d\tau} d\tau}{\bar{k}} \quad (5b)$$

The approximate equation for  $\overline{H}^2$  (eq. 7a) remains unchanged in form; one merely uses  $\overline{k}, \overline{\tau}, \overline{\tau}^2$  as defined here in place of the previously defined ones.

Let us now make a comparison between eq. (7a) and the corresponding one in CF-2848 X (eq. 5). In our notation the latter equation becomes ( $\overline{H}_W^2$  is Weinberg's second approximation for the Laplacian.)

$$\overline{H}_W^2 = \frac{\ln \overline{K}}{\overline{\tau}_f} (1 + \omega) \text{ with } \omega = \frac{\overline{\tau}_f - \overline{\tau}}{\overline{\tau}_f}. \quad (6b)$$

On the other hand since  $\overline{\tau} = \overline{\tau}_f(1 - \omega)$

$$\overline{H}_0^2 = \frac{\ln \overline{K}}{\overline{\tau}_f(1 - \omega)}. \quad (7b)$$

Therefore our first approximation,  $\overline{H}_0^2$ , is larger than Weinberg's second by the amount

$$\overline{H}_0^2 - \overline{H}_W^2 = \frac{\ln \overline{K}}{\overline{\tau}_f} \frac{\omega^2}{1 - \omega} \quad (8b)$$

and the ratio  $\overline{H}_0^2 / \overline{H}_W^2$  is:

$$\frac{\overline{H}_0^2}{\overline{H}_W^2} = \frac{1}{1 - \omega^2} > 1. \quad (9b)$$

We also note that our second approximation is larger than our first: i.e., that

$$\frac{\overline{\delta f_1^2}}{\overline{\delta f_w^2}} = \frac{1 + \overline{\tau} \overline{\delta f_0^2} \beta}{1 - \omega^2} > 1. \quad (10b)$$

Since the correction term in our second approximation, namely  $\overline{\tau} \overline{\delta f_0^2} \beta$  is always positive; this follows from the fact that  $\overline{\tau^2} \geq \overline{\tau}^2$ .

In one case considered in CF-2343 X,  $\omega \sim 0.40$ . Neglecting  $\omega^2/1-\omega$  in such a case causes a 19% error in  $\overline{\delta f_w^2}$ . In most of the piles considered there,  $\omega$  is much smaller and the error is only a few percent. In general the ratio between  $\overline{\delta f_0^2}$  and  $\overline{\delta f_w^2}$  increases as the  $\overline{\tau}$  decreases; that is, as the concentration increases or the slowing down power decreases.

(c) In CP-2381 consideration was given to a model in which absorption took place in narrow energy bands -- the discrete level model. For this model the equations (1), (3), (4) and (5) become

$$1 = \sum_1 \eta_1 (1 - P_1) \sum_{j=1}^{\infty} \int_{\tau_j}^{\tau_{j+1}} (\tau_1 - \tau') \overline{\delta f^2} f(\tau') d\tau' \prod_{k=i+1}^j P_k \quad (1c)$$

where  $P_k$  is the probability that a neutron escape capture in the  $k^{\text{th}}$  level on slowing down through the energy characteris-

tic of the level,  $\eta_l$  is the number of neutrons produced per neutron captured in the  $l^{\text{th}}$  level, and  $\bar{\tau}_l$  is the age at the  $l^{\text{th}}$  level;

$$\bar{k} = \sum_i k_i \text{ with } k_i = \eta_i (1 - P_i) \sum_{j=1}^{\infty} \int_{\tau_j}^{\tau_{j+1}} f(\tau) d\tau \prod_{k=1+1}^j P_k \quad (3c)$$

$$\bar{\tau} = \sum_i \eta_i (1 - P_i) \sum_{j=1}^{\infty} \int_{\tau_j}^{\tau_{j+1}} (\tau - \tau_i) f(\tau) d\tau \prod_{k=1+1}^j P_k / k \quad (4c)$$

$$\bar{\tau}^2 = \sum_i \eta_i (1 - P_i) \sum_{j=1}^{\infty} \int_{\tau_j}^{\tau_{j+1}} (\tau - \tau_i)^2 f(\tau) d\tau \prod_{k=1+1}^j P_k / k^2 \quad (5c)$$

If we specialize further for the moment, so that  $f(\tau) \neq 0$  only when  $\tau >$  any  $\tau_j$ , these equations simplify to

$$1 = \sum_i k_i e^{-\tau_i / \bar{\tau}} \int_0^{\infty} e^{-\tau / \bar{\tau}} f(\tau) d\tau \quad (1c')$$

$$\bar{k} = \sum_i k_i; \quad k_i = \eta_i (1 - P_i) \prod_{k>i} P_k \quad (3c')$$

$$\bar{\tau} = \sum_i k_i \int_0^{\infty} (\tau - \tau_i)^2 f(\tau) d\tau / k \quad (5c')$$

In CP-2881, (1c') was treated by substituting  $e^{-\bar{\tau}_0 \bar{\nu}^2}$  for  $\int_0^{\infty} e^{-\tau \bar{\nu}^2} f(\tau) d\tau$  where  $\bar{\tau}_0 = \int_0^{\infty} \tau f(\tau) d\tau$ .

The solution of

$$1 = \sum_1 k_i e^{-(\bar{\tau}_0 - \tau_i) \bar{\nu}^2} \quad (1c'')$$

is used there as the basic approximation for  $\bar{\nu}^2$ . Corrections are made by using better approximations for

$$\int_0^{\infty} e^{-\tau \bar{\nu}^2} f(\tau) d\tau.$$

This method of CP-2881, then, uses a different series of approximations from that in the present paper.

(d) The method of CP-2881 can also be applied in the case of continuous absorption under the assumption that  $f(\tau) \neq 0$  only for  $\tau$  sufficiently large that  $dp/d\tau = 0$ . Then eq. (1) becomes

$$1 = \int_0^{\infty} \eta(\tau) e^{\tau \bar{\nu}^2} \frac{dp}{d\tau} d\tau \int_0^{\infty} e^{-\tau' \bar{\nu}^2} f(\tau') d\tau' \quad (1d)$$

and the basic approximation gives

$$1 = \int_0^{\infty} \eta(\tau) e^{\tau \bar{\nu}^2} \frac{dp}{d\tau} d\tau e^{-\bar{\tau}_0 \bar{\nu}^2} \quad (1d')$$

Unlike the parallel equation in the discrete level case (1d') is not easily solvable, and approximate solution of (1d') leads

leads back to the general results of this report. To see the relation easily we note that the equivalent first approximation for

$$\int_0^{\infty} \eta(\tau) \frac{dP}{d\tau} e^{-\tau/k} d\tau \quad \text{is} \quad e^{-\tau_1/k} \bar{k}$$

where

$$\tau_1 = \frac{\int_0^{\infty} \tau \eta(\tau) \frac{dP}{d\tau} d\tau}{\bar{k}}$$

It is also easily verified that  $\bar{\tau} = \bar{\tau}_0 - \tau_1$ . Consequently we obtain

$$1 = \bar{k} e^{-\tau_1/k} \quad (1d'')$$

as in our general treatment in section (a).

It is, of course, possible to obtain the successive approximations to the

$$\int_0^{\infty} f(\tau) e^{-\tau/k} d\tau$$

just as in CP-2831 and similarly to get higher approximations to

$$\int_0^{\infty} \eta(\tau) e^{-\tau/k} \frac{dP}{d\tau} d\tau$$

The corrections are then separated into those arising from the higher moments of the fission spectrum and those arising from the moments of the absorption spectrum. The answer is subsumed

in the general result already given, but the separation is possible here because the fission and absorption spectra are assumed not to overlap.

(c) In carrying out the calculation of the critical size or "Laplacian" we need to have at our disposal  $f(\tau)$ ,  $P(\tau', \tau)$  and  $\eta(\tau)$ . For the determination of  $f(\tau)$ , we employ the fission spectrum as given in LA-200. This spectrum is for the fast fission of  $U^{235}$ , but we use it here for all energies of incident neutrons and for  $Pu^{239}$  merely because we have no better information and some indications that the fission spectra are very similar (cf. LA-84). For a Be moderated system we have used the age-energy relation computed from recent cross section data. (See forthcoming memorandum by M. L. Goldberger). LA-200 gives the fission spectrum in the form  $f(E)$ , the number of neutrons in unit energy interval about  $E$ , versus  $E$ . To transform to  $f(\tau)$ , the number of neutrons in unit age interval about  $\tau$ , we use the following relation:

$$f(\tau) = f(E) \frac{dE}{d\tau} = f(E)E \left[ 3 \frac{\sigma_{tr}}{\sigma_s} (\sigma_s \xi) \right] \quad (1e)$$

where  $\sigma_{tr}$  is the transport cross section per  $cm^3$ ,  $\sigma_s$ , the scattering cross section per  $cm^3$ , and  $\xi$ , the average logarithmic decrement per collision. In Fig. 1  $f(\tau)$  is plotted as a function of the Be age.

Wigner (C-1, CP-668) gives for the resonance escape probability,  $P(\tau', \tau)$ ,

$$P(\tau', \tau) = \exp \left\{ - \int_{\tau}^{\tau'} 3(\sigma_a)_{eff} \sigma_{tr} d\tau \right\} \quad (2e)$$

where  $(\sigma_a)_{\text{eff}} = \frac{\sigma_a}{1 + \sigma_a/\sigma_s}$  and  $\sigma_a$  is the absorption cross section per  $\text{cm}^3$ . It follows that  $dP(\mathcal{Z}^i, \mathcal{Z})/d\mathcal{Z}$  is given by

$$\frac{dP(\mathcal{Z}^i, \mathcal{Z})}{d\mathcal{Z}} = 3(\sigma_a)_{\text{eff}} \sigma_{\text{tr}} P(\mathcal{Z}^i, \mathcal{Z}). \quad (3e)$$

It is through  $(\sigma_a)_{\text{eff}}$  that the effect of various concentrations of fissionable isotopes in the reactor enters the calculation. As an example, consider Pu diluted in Be.

$$\sigma_a = N_{\text{Be}} \sigma_a^{\text{Be}} + N_{\text{Pu}} \sigma_a^{\text{Pu}} = N_{\text{Be}} \left( \sigma_a^{\text{Be}} + \frac{\sigma_a^{\text{Pu}}}{m} \right)$$

$$\sigma_s = N_{\text{Be}} \sigma_s^{\text{Be}} + N_{\text{Pu}} \sigma_s^{\text{Pu}} = N_{\text{Be}} \left( \sigma_s^{\text{Be}} + \frac{\sigma_s^{\text{Pu}}}{m} \right)$$

where  $\sigma_s^i, \sigma_a^i$  are the scattering and absorption cross sections respectively of "i" atoms,  $N_i$  is the number of "i" atoms/ $\text{cm}^3$ , and  $m = \frac{N_{\text{Be}}}{N_{\text{Pu}}}$ . (The atom concentration of Pu =  $1/(m+1)$ .)

$$(\sigma_a)_{\text{eff}} = \frac{\sigma_a}{1 + \sigma_a/\sigma_s} = \frac{N_{\text{Be}} \left( \sigma_a^{\text{Be}} + \frac{\sigma_a^{\text{Pu}}}{m} \right)}{1 + \left( \frac{m\sigma_a^{\text{Be}} + \sigma_a^{\text{Pu}}}{m\sigma_s^{\text{Be}} + \sigma_s^{\text{Pu}}} \right)}$$

We work normally in the range where  $\sigma_s^{\text{Pu}} \ll m\sigma_s^{\text{Be}}$  and  $m\sigma_a^{\text{Be}} \ll \sigma_a^{\text{Pu}}$ . If these conditions prevail,

$$(\sigma_a)_{\text{eff}} = \frac{N_{\text{Be}}}{m} \frac{\sigma_a^{\text{Pu}}}{1 + \frac{\sigma_a^{\text{Pu}}}{m\sigma_s^{\text{Be}}}} = \frac{N_{\text{Be}} \sigma_s^{\text{Be}}}{1 + m \frac{\sigma_s^{\text{Be}}}{\sigma_a^{\text{Pu}}}}$$

Similarly,

$$\sigma_{\text{tr}} = N_{\text{Be}} \sigma_{\text{tr}}^{\text{Be}} + N_{\text{Pu}} \sigma_{\text{tr}}^{\text{Pu}} = N_{\text{Be}} \left( \sigma_{\text{tr}}^{\text{Be}} + \frac{\sigma_{\text{tr}}^{\text{Pu}}}{m} \right)$$

$$\approx N_{\text{Be}} \sigma_{\text{tr}}^{\text{Be}}.$$

As the reactor is run the fission product absorption should be included in  $\sigma_a$ . This consideration is postponed until the change in critical size may be compared with that in breeding gain as the fission products accumulate.

Perhaps the greatest single inaccuracy in the calculations of resonance pile performance is the uncertainty of  $\eta(\mathcal{E})$ , the number of neutrons released per absorption in  $\text{Pu}^{239}$ .  $\eta_{49}(\mathcal{E})$  may be written as

$$\eta_{49}(\mathcal{E}) = \frac{\delta}{1 + \alpha(\mathcal{E})} \quad (4e)$$

where  $\delta$  is the number of neutrons released per fission (2.95 for  $\text{Pu}^{239}$ ) and  $\alpha$  is the ratio of radiative capture to fission cross section. The most recent data (LA-266) seems to require that  $\alpha(E)$  be negative for  $E > 1$  ev. Rather than accept negative  $\alpha$  values, we decided to use the values of  $\alpha$  as given by Weinberg (CF-2848) calculated from other data; Weinberg assumed that when  $\alpha$  went negative, it should be taken to be zero. His values of  $\eta$  are reproduced in Table 1. Note that for  $E > 20$ ,  $\alpha$  is zero and  $\eta = \delta$ .

Now that  $f(\mathcal{E})$ ,  $dP(\mathcal{E}; \mathcal{E})/d\mathcal{E}$ , and  $\eta(\mathcal{E})$  are known functions of  $\mathcal{E}$ , the critical size calculation is relatively simple. The calculation of  $\bar{k}$  is particularly easy if the absorption spectrum and fission spectrum do not overlap, for in this case,

$$\bar{k} = \int_0^{\mathcal{E}(20 \text{ ev})} \eta(\mathcal{E}) \frac{dP}{d\mathcal{E}} d\mathcal{E} + \{1 - P(20 \text{ ev})\} \eta_{\text{max}} \quad (5e)$$

Since the absorption is proportional to  $dP(\bar{\tau}, \tau)/d\tau$ , the separability of absorption and fission is, of course, apparent from the  $dP(\bar{\tau}, \tau)/d\tau$  and  $f(\bar{\tau})$  values. In this case, as has been pointed out previously

$$\bar{\tau} = \bar{\tau}_0 - \frac{\int_0^{\infty} \bar{\tau} \eta(\tau) \frac{dP}{d\tau} d\tau}{k} \quad (6e)$$

where  $\bar{\tau}_0$  is the arithmetic average of the age from fission to thermal. Unless the concentrations of metal are quite large (greater than one atom of Pu to 50 atoms of Be) separation obtains.

Detailed calculations have been carried out for Be-Pu systems for three values of  $m$  = number Be atoms/number Pu atoms. The results are summarized in Table 2.\* The calculations were carried out for a bare spherical reactor. Losses to Be were neglected.

## II. Breeding Characteristics

(a) In evaluating a pile for the purposes of producing new valuable isotopes, or of breeding, we wish to compute the average number of extra neutrons produced per destruction of valuable isotope in the pile; i.e., the number of neutrons available over and above the number needed to make the pile chain reacting. In these considerations, the number of neutrons produced on the average per absorption and per destruction of fissionable isotope in the reactor play central roles.

The number of neutrons absorbed in age interval  $d\tau$  about  $\tau$  in a bare reactor (i.e., no reflector) is proportional to

\* Table 2 is contained in CF-3490, p. 32.

$$\int_0^{\infty} e^{-(\bar{\tau} - \tau_1)k^2} f(\tau_1) \frac{dP(\tau_1, \bar{\tau})}{d\tau} d\tau_1. \quad (1a)$$

Then the total number of neutrons produced is proportional to

$$\int_0^{\infty} \eta(\bar{\tau}) \int_0^{\infty} e^{-(\bar{\tau} - \tau_1)k^2} f(\tau_1) \frac{dP(\tau_1, \bar{\tau})}{d\tau} d\tau_1 d\bar{\tau}. \quad (2a)$$

and the total number of absorptions is proportional with the same factor to

$$\int_0^{\infty} \int_0^{\infty} e^{-(\bar{\tau} - \tau_1)k^2} f(\tau_1) \frac{dP(\tau_1, \bar{\tau})}{d\tau} d\tau_1 d\bar{\tau}. \quad (3a)$$

Thus we have the number of neutrons produced on the average per absorption of any kind  $\bar{\eta}$ :

$$\bar{\eta} = \frac{\int_0^{\infty} \eta(\bar{\tau}) \int_0^{\infty} e^{-(\bar{\tau} - \tau_1)k^2} f(\tau_1) \frac{dP(\tau_1, \bar{\tau})}{d\tau} d\tau_1 d\bar{\tau}}{\int_0^{\infty} \int_0^{\infty} e^{-(\bar{\tau} - \tau_1)k^2} f(\tau_1) \frac{dP(\tau_1, \bar{\tau})}{d\tau} d\tau_1 d\bar{\tau}}$$

Using eq. (1a) of Section I,

$$\bar{\eta} = \frac{1}{\int_0^{\infty} \int_0^{\infty} e^{-(\bar{\tau} - \tau_1)k^2} f(\tau_1) \frac{dP(\tau_1, \bar{\tau})}{d\tau} d\tau_1 d\bar{\tau}} \quad (4a)$$

Similarly, if  $u(\bar{\tau})$  is the utilization of those absorbed in  $d\bar{\tau}$  at  $\bar{\tau}$  in destroying fissionable isotopes, the number of neutrons produced on the average per destruction,  $\bar{\eta}$ , is

$$\bar{\eta} = \frac{1}{\int_0^{\infty} u(\tau) \int_{\tau}^{\infty} e^{-(\tau-\tau')\beta^2} f(\tau') \frac{dP(\tau', \tau)}{d\tau} d\tau' d\tau} \quad (5a)$$

We are now in position to compute the fertility,  $F$ , of the system which may be defined as the number of new valuable isotopes produced per destruction of fissionable isotope in the reactor. If  $\bar{u}_0$  is the average number of new isotopes produced for each neutron escaping from the reactor, the contribution to  $F$  from outside the reactor is

$$F_0 = \frac{\bar{u}_0 \int_0^{\infty} \{\bar{\eta}(\tau) - 1\} \int_{\tau}^{\infty} e^{-(\tau-\tau')\beta^2} f(\tau') \frac{dP(\tau', \tau)}{d\tau} d\tau' d\tau}{\int_0^{\infty} u(\tau) \int_{\tau}^{\infty} e^{-(\tau-\tau')\beta^2} f(\tau') \frac{dP(\tau', \tau)}{d\tau} d\tau' d\tau} \quad (6a)$$

This may be rewritten using  $\bar{\eta}$  and  $\bar{\eta}$  as follows: Since

$$\frac{\bar{\eta}}{\bar{\eta}} = \frac{\text{no. neutrons absorbed}}{\text{no. destructions}}$$

and  $\bar{\eta} - 1 = \text{no. extra neutrons produced per absorption,}$

$$F_0 = \bar{u}_0 \frac{\bar{\eta}}{\bar{\eta}} (\bar{\eta} - 1). \quad (7a)$$

It is easily seen that this may be written in the more familiar form

$$F_0 = \bar{u}_0 (\bar{\eta} - 1 - L) \quad (8a)$$

where

$$L = \frac{\frac{\bar{\eta}}{\bar{\eta}} - 1 = \frac{\int_0^{\infty} [1 - u(\tau)] \int_{\tau}^{\infty} e^{-(\tau - \tau')\lambda^2} f(\tau') \frac{dP(\tau', \tau)}{d\tau} d\tau' d\tau}{\int_0^{\infty} u(\tau) \int_{\tau}^{\infty} e^{-(\tau - \tau')\lambda^2} f(\tau') \frac{dP(\tau', \tau)}{d\tau} d\tau' d\tau}}{\quad} \quad (9a)$$

L is the "parasitic" absorption in the pile per destruction. Possibly there may be a contribution to F in this absorption provided some valuable absorber be put inside the reactor.

If  $\bar{u}_p$  is the average number of valuable isotopes produced per parasitic capture in the reactor

$$F = F_0 + \bar{u}_p L$$

$$= \bar{u}_0 \left( \frac{\bar{\eta}}{\bar{\eta}} - \frac{\bar{\eta}}{\bar{\eta}} \right) + \bar{u}_p \left( \frac{\bar{\eta}}{\bar{\eta}} - 1 \right).$$

$\bar{u}_p L$  may be brought into the general scheme by

$$\bar{u}_p L = \frac{\int_0^{\infty} u_p(\tau) (1 - u(\tau)) \int_{\tau}^{\infty} e^{-(\tau - \tau')\lambda^2} f(\tau') \frac{dP}{d\tau} d\tau' d\tau}{\int_0^{\infty} u(\tau) \int_{\tau}^{\infty} e^{-(\tau - \tau')\lambda^2} f(\tau') \frac{dP}{d\tau} d\tau' d\tau}$$

with  $u_p(\tau)$  the number of valuable isotopes produced per parasitic absorption at age  $\tau$ .

(b) It is appropriate at this point to say a few words about the relative magnitudes of the three average "reproduction factors"  $\bar{k}$ ,  $\bar{\eta}$ , and  $\frac{\bar{\eta}}{\bar{\eta}}$ . It is obvious from the definitions that  $\frac{\bar{\eta}}{\bar{\eta}} \geq \bar{\eta}$ , the equality holding only if  $u(\tau) = 1$ . If  $\eta(\tau) =$

const =  $\eta_0$ , then  $\bar{k} = \eta_0 = \bar{\eta} \leq \overline{\eta}$ . On the other hand if  $\eta$  is not constant,  $\bar{\eta}$  is a weighted average over  $\eta$  including leakage and  $\bar{k}$  is a weighted average over  $\eta$  excluding leakage. Consequently, if  $\eta$  increases with energy,  $\bar{\eta} > \bar{k}$  and  $\eta_{\max} > \bar{\eta} > \bar{k}$  where  $\eta_{\max}$  is the maximum number of neutrons released per absorption in the fissionable isotope.

(c) We shall write down for the cases of monoenergetic fission neutrons, the discrete level model, and separable fission and absorption. The equations analogous to those of (a):

$$1. f(z) = \delta(z - z_f)$$

$$\bar{\eta} = \frac{1}{\int_0^{z_f} e^{-(z-z_f)/\lambda} \frac{dP(z_f, z)}{dz} dz}$$

$$\overline{\eta} = \frac{1}{\int_0^{z_f} u(z) e^{-(z-z_f)/\lambda} \frac{dP(z_f, z)}{dz} dz}$$

## 2. Discrete Level Model

$$\bar{\eta} = \frac{1}{\sum_i (1 - P_i) \sum_{j=1}^{\infty} \int_{z_j}^{z_{j+1}} e^{-(z_i - z_j)/\lambda} f(z_i) dz_i \prod_{k=i+1}^j P_k}$$

$$\overline{\eta} = \frac{1}{\sum_i u_i (1 - P_i) \sum_{j=1}^{\infty} \int_{z_j}^{z_{j+1}} e^{-(z_i - z_j)/\lambda} f(z_i) dz_i \prod_{k=i+1}^j P_k}$$

where  $u_1$  is the utilization at the 1<sup>th</sup> level.

If  $f(\tau) \neq 0$  only when  $\tau$  is greater than any  $\tau_j$ .

$$\bar{\eta} = \frac{1}{\sum_i (1 - P_i) \prod_{k>i} P_k} \int_0^{\infty} \tau^{-2} e^{-\tau^{-2}} f(\tau) d\tau$$

$$\bar{\eta} = \frac{1}{\sum_i u_i (1 - P_i) \prod_{k>i} P_k} \int_0^{\infty} \tau^{-2} e^{-\tau^{-2}} f(\tau) d\tau$$

3. Continuous absorption,  $f(\tau) \neq 0$  only when  $dP(\tau_1, \tau)/d\tau = 0$

$$\bar{\eta} = \frac{1}{\int_0^{\infty} \tau^{-2} \frac{dP}{d\tau} d\tau \int_0^{\infty} \tau^{-2} e^{-\tau^{-2}} f(\tau_1) d\tau_1} = \frac{\int_0^{\infty} \tau^{-2} \frac{dP}{d\tau} d\tau}{\int_0^{\infty} \tau^{-2} \frac{dP}{d\tau} d\tau}$$

$$\bar{\eta} = \frac{1}{\int_0^{\infty} u(\tau) \tau^{-2} \frac{dP}{d\tau} d\tau \int_0^{\infty} \tau^{-2} e^{-\tau^{-2}} f(\tau_1) d\tau_1} = \frac{\int_0^{\infty} \tau^{-2} \frac{dP}{d\tau} d\tau}{\int_0^{\infty} u(\tau) \tau^{-2} \frac{dP}{d\tau} d\tau}$$

(NOTE: Through the whole of section II as at present written, the assumption has been that the reactor is essentially bare; i.e., that the blanket in which peripheral production takes place does not act as a reflector. The modifications which must be introduced when this blanket is a reflector have been consider-

ed from a multigroup point of view in CP-2881 but we have currently engaged in some slightly different methods particularly for the case of a blanket in which slowing down does not take place. Our results so far are not great or astounding. In any more or less final version of this report, we expect to have something, however, which belongs at this point.)

### III. Space Variable Piles

Since heat production is most intense at the center of a pile, the temperature rise in uniformly spaced, equal area cooling channels running parallel to the axis of a cylindrical pile is not uniform from center to edge of the pile. Such an arrangement demands more coolant than is actually necessary or even desirable since not only do the losses increase with added coolant but the utilization of the available power is certainly not very efficient. In addition, it might be desirable from an engineering point of view to have the temperature of the coolant uniform over the pile exit cross section.

In order to achieve more efficient performance, it is clear that the properties of the pile must be space variable. The exact spatial variations necessary in one case will be discussed later. This whole problem is of particular importance in resonance and fast breeders where one wishes to keep the losses down and decrease the doubling time. We shall consider therefore, a method by which the properties of such a pile may be determined.

Our starting point is the Fermi equation (CP-1662) with all cross sections/cm<sup>3</sup> having spatial variations over the pile. We shall assume that the spatial variation of the slowing down power/cm<sup>3</sup> is the same as that of the absorption cross section/cm<sup>3</sup>; the spatial variation of the transport cross section (and hence of the diffusion coefficient) need not be the same; but the spatial variation of all three is independent of energy.

Written in its most general form, the Fermi equation is

$$\text{div } D \text{ grad } \varphi + \frac{\partial(S\varphi)}{\partial x} - \varphi a + \left[ \int_0^{\infty} \varphi(x', \vec{r}) a(x', \vec{r}) \eta(x') dx' \right] f(x) = 0 \quad (1)$$

where  $\varphi$  = neutron density times velocity;  $x = \ln(E/E_{\text{thermal}})$ ;  $a$  = absorption cross section/cm<sup>3</sup>;  $S = (\sum N_i \sigma_s^i)_{\xi_1}$  = slowing down power/cm<sup>3</sup>;  $D = 1/3 \sum N_i \sigma_{tr}^i$ ;  $\eta$  = no. of neutrons released per absorption of any kind of neutron of "energy"  $x$ ;  $f(x)$  is the normalized fission spectrum ( $\int_0^{\infty} f(x) dx = 1$ ). According to our assumptions we write

$$D = D_0(x) \gamma(\vec{r}), \quad a(x, \vec{r}) = a_0(x) \lambda(\vec{r}), \quad S = S_0(x) \lambda(\vec{r})$$

where  $\lambda(\vec{r})$  and  $\gamma(\vec{r})$  are dimensionless functions of position.

Now let us try a solution (1) of the form  $\varphi = \varphi_S(\vec{r}) X(x)$ . We find

$$\frac{\text{div} [\gamma \text{ grad } \varphi_S]}{\lambda \varphi_S} + \frac{1}{D_0 X} \frac{\partial(S_0 X)}{\partial x} - \frac{a_0}{D_0} + \frac{A f(x)}{D_0 X} = 0 \quad (2)$$

where  $A = \int_0^{\infty} X(x') a_0(x') \eta(x') dx'$ . Since the first term is a function of space only and the others are functions of  $x$  only,

we must have

$$\operatorname{div} [\gamma \operatorname{grad} \varphi_s] + \overline{\gamma}^2 \lambda \varphi_s = 0 \quad (3a)$$

and

$$\frac{\partial (S_0 X)}{\partial x} - \left( \frac{a_0 + D_0 \overline{\gamma}^2}{S_0} \right) S_0 X + A f(x) = 0 \quad (3b)$$

We can easily verify by differentiation that the solution of (3b) is

$$S_0(x)X = A \int_x^\infty e^{-\int_x^{x'} \frac{a_0(x'') + D_0(x'') \overline{\gamma}^2}{S_0(x'')} dx''} f(x') dx'. \quad (4)$$

By multiplying both sides by  $\gamma(x)a_0(x)/S_0(x)$  and integrating over  $x$ , we find for the characteristic equation which determines  $\overline{\gamma}^2$

$$1 = \int_0^\infty \gamma(x) \frac{a_0(x)}{S_0(x)} \int_x^\infty e^{-\int_x^{x''} \frac{a_0(x''') + D_0(x''') \overline{\gamma}^2}{S_0(x''')} dx'''} f(x') dx'. \quad (5)$$

If we now define

$$\tilde{z}(x) = \int_0^x \frac{D_0(x)}{S_0(x)} dx$$

and

$$P(x', x) = \exp \left\{ - \int_x^{x'} \frac{a_0(x'')}{S_0(x'')} dx'' \right\}$$

We can easily transform eq. (5) into

$$1 = \int_0^\infty \gamma(\tilde{z}) \int_{\tilde{z}}^\infty e^{-(\tilde{z}' - \tilde{z}) \overline{\gamma}^2} f(\tilde{z}') \frac{dP(\tilde{z}', \tilde{z})}{d\tilde{z}} d\tilde{z}' d\tilde{z}. \quad (6)$$

This equation is formally identical with eq. (1a) of section I. Consequently, the methods which were developed there for its solution may be applied here. We also note that  $\tau(x)$  which is the  $\tau$  employed in eq. (6) is the Fermi age in a medium for which the ratio  $\lambda/\delta = 1$ .

Because in the actual pile the appropriate age changes from point to point as the ratio  $\lambda/\delta$  changes, the meaning of  $\overline{\mathcal{H}}^2$  here is not the same as it has been in previous sections of this report. Previously the spatial variation of neutron density throughout the pile was determined from the simple equation  $\Delta \varphi + \overline{\mathcal{H}}^2 \varphi = 0$ . In this section, the spatial variation must be determined from the more complicated equation (3a).

We now ask: for what value of  $\overline{\mathcal{H}}^2$  used in the simple equation we obtain the same results for critical size which are gotten from solving (3a) exactly? This value of  $\overline{\mathcal{H}}^2$  in the simple equation we shall call  $\overline{\mathcal{H}}_0^2$ . Let us write the simple equation and a slightly modified form of (3a). They are

$$\Delta \varphi_0 + \overline{\mathcal{H}}_0^2 \varphi_0 = 0 \quad (7)$$

and

$$\Delta \varphi + \text{grad } \ln \delta \cdot \text{grad } \varphi + \overline{\mathcal{H}}^2 \frac{\lambda}{\delta} \varphi = 0. \quad (3a')$$

Multiplying eq. (7) by  $\varphi$  and eq. (3a') by  $\varphi_0$  and subtracting, we obtain

$$\text{div} \left[ \varphi_0 \text{grad } \varphi - \varphi \text{grad } \varphi_0 \right] + \varphi_0 \text{grad } \ln \delta \cdot \text{grad } \varphi - \left( \overline{\mathcal{H}}^2 \frac{\lambda}{\delta} - \overline{\mathcal{H}}_0^2 \right) \varphi_0 \varphi = 0. \quad (8)$$

By vector manipulation of the second term, eq. (8) may be modified to give

$$\operatorname{div} [\varphi_0 \operatorname{grad} \varphi - \varphi \operatorname{grad} \varphi_0 + \varphi_0 \varphi \operatorname{grad} \ln \gamma] +$$

$$(\overline{\mathcal{H}}^2 \frac{\lambda}{\gamma} - \overline{\mathcal{H}}_0^2 - \Delta \ln \gamma - \operatorname{grad} \ln \gamma \cdot \operatorname{grad} \ln \varphi_0) \varphi_0 \varphi = 0 \quad (8a)$$

By integrating eq. (8a) over the volume of the reactor and applying Gauss' lemma to the first term, we obtain for  $\overline{\mathcal{H}}_0^2$

$$\overline{\mathcal{H}}_0^2 = \frac{\int \varphi_0 \varphi \left[ \overline{\mathcal{H}}^2 \frac{\lambda}{\gamma} - \Delta \ln \gamma - \operatorname{grad} \ln \gamma \cdot \operatorname{grad} \ln \varphi_0 \right] dV}{\int \varphi_0 \varphi dV} \quad (9)$$

since both  $\varphi_0$  and  $\varphi$  are zero at the surface of the reactor.

It is, of course, true that the value of  $\overline{\mathcal{H}}_0^2$  is independent of any arbitrary choices which may apparently have entered the calculation. There are really only two such choices which we have made. They are the values picked for  $D_0$  and  $S_0$  at some given energy. We wish now to verify that the choices of  $D_0$  and  $S_0$  do not influence the value of  $\overline{\mathcal{H}}_0^2$  as computed in eq. (9). In order to do this, we must show two things; that the  $\operatorname{grad} \ln \gamma$  and  $\overline{\mathcal{H}}^2 \frac{\lambda}{\gamma}$  are independent of our choices. From the definition of  $\gamma$  it follows that  $\operatorname{grad} \ln \gamma = \operatorname{grad} \ln D(x,r)$ . This demonstrates the first of our two conditions. In order to show that  $\overline{\mathcal{H}}^2 \frac{\lambda}{\gamma}$  is independent of our choices, we note that a different choice of  $D_0/S_0$  will merely change  $\frac{\lambda}{\gamma}$  by a scale factor, say,  $\psi$ . From eq. (5), it then follows that the value of  $\overline{\mathcal{H}}_0^2$  must be multiplied by the factor  $1/\psi$ .

Consequently, the value of  $\frac{\lambda^2}{\gamma^2}$  is exactly the same as before.

In order to use eq. (9) it is necessary to have the functions  $\varphi_0$  and  $\varphi$ . For simple geometries  $\varphi_0$ , the solution of eq. (7) is usually well-known. In order to find  $\varphi$  exactly, we would have to solve eq. (3a'). However, to get a first approximation for the value of  $\frac{\lambda^2}{\gamma^2}$ , we may adopt the usual perturbation theory assumption that  $\varphi$  may be replaced by  $\varphi_0$ .

This section was introduced by some qualitative considerations which pointed out the value of piles with specially varying properties. The spacial variation was introduced to decrease losses to the coolant and reduce the critical mass.

As an example of the type of calculation that would be necessary for a real pile design, consider the following situation. We shall cool the pile by means of equal diameter channels through which a coolant flows; we keep the pressure drop the same over the pile (hence the flow velocity in each channel is the same) and wish to arrange the channels in such a way that the temperature rise in each one is the same. Let

$$q(\vec{r}) = \frac{\text{volume of voids}}{\text{volume of solid material}}$$

a quantity which is a measure of the spacing of the channels over the pile. To keep the temperature rise the same in all streams, the streams must be spaced so that the coolant volume  $V/(1+q)$  be proportional to the number of fissions per sec per unit volume,  $F$ . On the other hand, if the flux is  $\varphi$ ,  $F$

is proportional to  $\varphi/(1 + \alpha)$ . Thus we may write

$$\frac{\alpha}{1 + \alpha} : \frac{\varphi}{1 + \alpha} = \frac{\alpha_c}{1 + \alpha_c} : \frac{\varphi_c}{1 + \alpha_c}$$

where  $\alpha_c$  and  $\varphi_c$   
are the values of  
 $\alpha$  and  $\varphi$  at the  
center of the pile.

$$\alpha = \alpha_c \frac{\varphi}{\varphi_c}$$

In practice, in a cylindrical pile for example, the fraction of cooling ducts may be varied as a function of the radial distance from the axis of the pile but not as a function of the distance away from the center parallel to the pile axis. Consequently, in place of  $\alpha = \alpha_c \frac{\varphi}{\varphi_c}$ , we should use

$$\alpha = \alpha_c \frac{\varphi}{\varphi_c} \cos(\delta x) = \alpha_c \varphi_r$$

(With the appropriate value of  $\delta$ ,  $\cos(\delta x)$  gives the longitudinal variation of  $\varphi$  in a cylindrical pile.)  $\varphi_r$  is the radial variation of  $\varphi$  normalized to one on the axis.

Suppose we employ a coolant which does negligible slowing down and has negligible absorption but whose scattering cross section/cm<sup>3</sup> is the same as the rest of the reactor material.

In this case  $\delta$  is constant and

$$\frac{\lambda}{\delta} = \frac{1}{1 + \alpha_c \varphi_r}$$

when  $\nabla^2$  is the negative laplacian for a pile with no coolant.

Substituting in eq. (9) we find for  $\nabla^2 \varphi_c$ :

$$\overline{H}_0^2 = \overline{H}^2 \frac{\int_0^{2.4048} J_0(z) \left[ \varphi_z / (1 + \alpha_c \varphi_z) \right] z dz}{\int_0^{2.4048} J_0(z) \varphi_z z dz}$$

with  $\varphi_z = \varphi_\rho$  and  $z = \beta\rho$  such that  $z = 2.4048$  when  $\rho =$  the radius of the cylinder. If  $\alpha_c \ll 1$ , we obtain

$$\overline{H}_0^2 = \overline{H}^2 \left[ 1 - \alpha_c \frac{\int_0^{2.4048} J_0(z) \varphi_z^2 z dz}{\int_0^{2.4048} J_0(z) \varphi_z z dz} + \dots \right].$$

Since  $\varphi_z \geq J_0(z)$ , if we substitute  $J_0(z)$  for  $\varphi_z$  we over-estimate  $\overline{H}_0^2$ . Physically this follows from the fact that using  $J_0(z)$  introduces less than the requisite amount of cooling into the pile. For small values of  $\alpha_c$ , however,  $\varphi_z \rightarrow J_0(z)$  so that when  $\alpha_c \ll 1$

$$\overline{H}_0^2 = \overline{H}^2 \left[ 1 - \alpha_c \frac{\int_0^{2.4048} [J_0(z)]^3 z dz}{\int_0^{2.4048} [J_0(z)]^2 z dz} + \dots \right].$$

On evaluating the integrals we obtain

$$\overline{H}_0^2 = \overline{H}^2 \left[ 1 - .725 \alpha_c + \dots \right].$$

If the pile were cooled with the same distribution of coolant everywhere which is required at the center the value of  $\overline{H}^2$  would be,  $\overline{H}_c^2$ :

$$\overline{H}_c^2 = \frac{\overline{H}^2}{1 + \alpha_c} = \overline{H}^2 \left[ 1 - \alpha_c + \dots \right]$$

and

$$\overline{\rho}_o^2 = \overline{\rho}_c^2 [1 + .275 \alpha_c - \dots]$$

On the other hand, the average density of standard reactor material in the space variable pile is

$$\frac{2}{(2.4048)^2} \int_0^1 \frac{z dz}{1 + \alpha_c \psi_z}$$

as compared with  $1/(1 + \alpha_c)$  for the uniformly cooled pile so that the ratio of standard reactor material in the two cases is

$$\frac{M_o}{M_c} = 1 + .155 \alpha_c \dots\dots$$

To the same approximation

$$\frac{M_o}{M} = 1 + .655 \alpha_c$$

and

$$\frac{M_c}{M} = 1 + .500 \alpha_c$$

Physically the reason that  $M_c < M_o$  is that the coolant which is removed in the case of  $M_o$  but present for  $M_c$  acts as a reflector increasing the efficiency of the central region of the pile. In  $M_o$  standard reactor material is, therefore, added to compensate the reflector effect lost on removing the coolant from the outside regions of  $M_o$ .

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